Pricing Corporate Bonds

- Bonds rated by rating agencies such as Moody's and S&P
- Collect data on actively traded bonds
- Calculate a generic zero-coupon yield curve for each credit rating category
- Value a newly issued bond using the zero-coupon yield curve for that category
- Higher yield on a corporate bonds over that of a risk free bond is compensation for possible losses from default (Liquidity is also a factor)
- Use treasury curve or LIBOR zero curve as the risk free zero curve

Computing Zero-Coupon Yield Curves

- **$n-$year Zero Rate $r(n)$:** $e^{n\,r(n)}$ is the value of 1 unit of currency invested at the risk free rate for n years
- **Pricing a Coupon yielding Bond**

Maturity	Zero Rate
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Table 1: Treasury Zero Rates

Present value of two year treasury bond with principal 100 that pays coupon at 6% per annum semiannually

$$3e^{-0.05\times0.5} + 3e^{-0.058\times1.0} + 3e^{-0.064\times1.5} + 103e^{-0.068\times2.0} = 98.39$$

- **Bond Yield:** The yield on a coupon-bearing bond is the discount rate that equates the cash flows on the bond to its market value

$$3e^{-y\times0.5} + 3e^{-y\times1.0} + 3e^{-y\times1.5} + 103e^{-y\times2.0} = 98.39 \qquad \Rightarrow \qquad y = 6.76\%$$

Determining Treasury Zero Rates

Bond principal	Maturity	Annual coupon	Bond price
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0

Table 2: Prices of Coupon Yielding Bonds

$$100 = 97.5e^{r(0.25)\times 0.25} \qquad \Rightarrow \qquad r(0.25) = 10.127\%$$

$$r(0.5) = 10.469\%, \qquad r(1.0) = 10.536\%$$

$$4e^{-0.10469\times 0.5} + 4e^{-0.10536\times 1.0} + 104e^{-r(1.5)\times 1.5} = 96 \qquad \Rightarrow \qquad r(1.5) = 10.681\%$$

Probability of Default: Quick Estimate

- $y(T)$ $(y^*(T))$: Yield on a $T-$year corporate (risk-free) zero-coupon bond
- $Q(T)$: Probability that corporation will default between time zero and time T
- **Recovery Rate R** Proportion of claimed amount received in the event of default
- **Value of the Bond**

$$[Q(T) \times 100R + (1 - Q(T)) \times 100]e^{-y^*(T)T} = 100e^{-y(T)T}$$

$$e^{-y^*(T)T} - e^{-y(T)T} = Q(T)(1 - R)e^{-y^*(T)T}$$

$$Q(T) = \frac{1 - e^{-[y(T)-y^*(T)]T}}{1 - R}$$

- $[y(T) - y^*(T)]$: **Spread** $Q(T)$: **Risk-Neutral Probability**
- $y(T)$: Not observed Claim amount = Face Value + Accrued Interest

Probability of Default: More Realistic Assumptions

- N coupon bearing bonds issued by the corporation being considered or by another corporation with roughly same probabilities of default
- Bond maturity times $t_1 < t_2 < \cdots < t_n$
- $q(t)$: Default probability density
- B_j : Price of the j^{th} bond
- G_j : Price of a risk free bond with same cash flow as the j^{th} bond
- $F_j(t)$: Forward price of the j^{th} bond for a forward contract maturing at time $t < t_j$ assuming bond is default free
- $v(t)$: present value of 1 received at time t with certainty
- $R_j(t)$: Recovery rate for default of j^{th} bond at time $t < t_j$

$$G_j - B_j = \int_0^{t_j} q(t)v(t)F_j(t)(1 - R_j(t))dt$$

- $q(t) = \sum_{i=1}^{N} q_i 1_{\{t_{i-1}<t\leq t_i\}}$

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t)F_j(t)(1 - R_j(t))dt$$

$$q_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i\beta_{ij}}{\beta_{jj}}$$

- Determine the q_j's inductively
- Results can be extended to the case when recovery and interest rates are random but mutually independent and independent of default events
- A-Rated bond with 30% recovery rate and yield 70 basis points above the risk-free rate will give a p.d. of 5%. Historical default probability is only 0.57%.
- Credit ratings are revised relatively infrequently

Credit Derivatives

Credit Default Swaps

- CDS - Contract that provides Insurance against the risk of a default
- Total *par* value of the bond is called *notional principal*
- Cash settlement - $(100 - Z)\%$ of notional principal, where Z is the mid market price some specified number of days after the credit event
- Physical delivery - Deliver the bonds in exchange for par value

Example: Consider a 5 yr. CDS with notional principal 100 million entered on March 1, 2002. Buyer pays 90 basis points annually.

No Default: Buyer pays $900,000$ on March 1 of each year from 2003 to 2007

Default event: Credit event on September 1, 2005. Buyer sells 100 million worth of bonds and receives 100 million and pays the accrued payment amount (approx. $450,000$)

Example: CDS swaps allow companies to manage their credit risk actively.

Suppose a bank has several millions dollars of loans outstanding to Enron on Jan. 2001.

1. Buy a 100 million 5-yr. CDS on Enron from market for 135 BP or 1.35 million per year. This would shift part of the bank's credit exposure to Enron.

   OR

2. Exchange part of the exposure to a company in a totally different industry

   Sell a five year 100 million CDS on Nissan for 1.25 million

   Net Cost $= 100,000$ per year

Valuation of CDS

- Assume notional to be 1
- Assume that default events, interest rates and recovery rates are independent
- T : Life of credit default swap in years
- q_t : Risk neutral probability of default at time t
- $R(t)$: Recovery rate when default happens at time t
- $u(t)$: present value of payments made at the rate of 1 per year on payment dates between 0 and t
- $v(t)$: present value of 1 received at time t
- w : Payments per year made by CDS buyer
- s : Value of w that causes the CDS to have a value of zero
- π : The risk neutral probability of no credit event during the life of the swap
- $A(t)$: Accrued interest on the reference obligation at time t as percent of face value

Valuation of CDS

$$\int_0^T [1 - R(t)(1 + A(t))]q(t)v(t)dt - w\int_0^T q(t)u(t)dt + w\pi u(T) = 0$$

Credit Default Swap Spread

$$s = \frac{\int_0^T [1 - R(t)(1 + A(t))]q(t)v(t)dt}{\int_0^T q(t)u(t)dt + w\pi u(T)}$$

Implying Default Probabilities from CDS Swaps

- Suppose CDS spreads for maturities $t_1, t_2, \ldots, t_n$ are $s_1, s_2, \ldots, s_n$
- Assume default probability density $q(t) = \sum_{i=1}^{N} q_i 1_{\{t_{i-1}<t\leq t_i\}}$

$$s_i = \frac{\sum_{k=1}^{i} q_k \int_{t_{k-1}}^{t_k} [1 - R(t)(1 - A_i(t))v(t)]dt}{\sum_{k=1}^{i} q_k \int_{t_{k-1}}^{t_k} u(t)dt + u(t_i)[1 - \sum_{k=1}^{i} q_k(t_k - t_{k-1})]}$$

$$\delta_k = t_k - t_{k-1}, \qquad \alpha_k = \int_{t_{k-1}}^{t_k} (1 - R(t))v(t)dt$$

$$\beta_{k,i} = \int_{t_{k-1}}^{t_k} A_i(t)R(t)v(t)dt, \qquad \gamma_k = \int_{t_{k-1}}^{t_k} u(t)dt$$

$$q_i = \frac{s_i u(t_i) + \sum_{k=1}^{i-1} q_k[s_i(\gamma_k - u(t_i)\delta_k) - \alpha_k + \beta_{k,i}]}{\alpha_i - \beta_{i,i} - s_i(\gamma_i - u(t_i)\delta_i)}$$

Basket Credit Default Swaps

- Number of reference entities
- **Add-up** basket CDS - provides a payoff when any of the reference entities default $\equiv$ portfolio of CDS, one for each entity
- **First-to-default** basket CDS - payoff only when the first reference entity defaults
- k^{th}**-to-default** CDS - payoff only when the k^{th} default happens
- **Modeling Dependencies**

Modeling Dependencies via Copulas

- **Joint distribution of default times**

$$F(t_1, t_2, \ldots, t_n) = C(F_1(t_1), \ldots, F_n(t_n))$$

- **Gaussian Copula**

$$C(u_1, \ldots, u_n) = N_n(N^{-1}(u_1), \ldots, N^{-1}(u_n); \Gamma)$$

- **Simulation:** Generate a realization of n correlated normal variables $Z_1, \ldots, Z_n$ from $N_n(\cdot; \Gamma)$. Obtain default times $\tau_i = F_i^{-1}(N(Z_i))$

Value Swaps using Monte-Carlo Simulations

Dynamic Intensity Models

- Reduced Form Model
- **Basic Affine or Intensity Process**

$$d\lambda(t) \;=\; \kappa(\theta - \lambda(t))\, dt + \sigma\sqrt{\lambda(t)}\, dB(t) + \Delta J(t)$$

- N firms in portfolio, S sectors
- **Correlated defaults** $\quad \lambda_i = X_i + a_i\, X_{s(i)} \; + \; b_i\, X_c$
  $X_{s(i)}, X_i, X_c$ independent BAP representing the **sectoral risk**, the **idiosyncratic risk** and **common risk**
- X_p BAP with parameters $(\kappa, \theta_p, \sigma, \mu, \ell_p)$, $\quad p = i, s(i), c,$
  $i = 1, \ldots, N,$ $\quad s(i) \in S.$
- $\lambda_i :$ BAP
- Simulate realizations of $\lambda_i(\cdot)$ and hence the compensator process
  $\Lambda_i(t) = \int_0^t \lambda_i(u) du$
- Simulate independent unit mean exponential random variables $E_1, \ldots, E_n$
- $\tau_i = \min\{t; \Lambda_i(t) = E_i\}$ if $\Lambda_i(T) < E_i$.

Calibrating the Model

- **Data:** Default Probabilities, Default Correlations, Yield Spreads
- **Long Run Mean:** $\sum_p(\theta_p + \ell_p\mu/\kappa)$
- **Multi-Issuer Default:** If X_c jumps, all λ_i jump and if $X_{s(i)}$ jumps, default intensities of all obligors in sector $s(i)$ jump
- Explicit formulas for default correlations
- Conditional survival probability

$$p_i(t,s) = \exp\left[\alpha(s) + \sum_p \beta_p(s) X_p(t)\right]$$

Coefficients can be obtained explicitly (see Appendix of Credit Risk by Duffie and Singleton)

Structural Credit Models

- **Firm's market value** V follows a geometric Brownian motion

$$\frac{dV_t}{V_t} = \mu \, dt + \sigma \, dW_t; \qquad V_0 > 0$$

$$V_t = V_0 e^{mt + \sigma W_t}, \qquad m = \mu - \frac{1}{2}\sigma^2$$

- Firm is financed by **equity** and a **zero coupon bond** with face value K with maturity date T
- Bond investors have absolute priority in case of default
- Firm **defaults** if $\tau = \min(\tau_1, \tau_2) \leq T, \;\; \tau_2 = T$ if $V_T < K$ and ∞ otherwise

$$\tau_1 = \inf\{t \geq 0 : V_t < D\}$$

- $L = \frac{K}{V_0}$: initial leverage ratio

- **Default Probability** **Apply Girsanov**

$$\begin{aligned} p(T) &= 1 - P[M_T > D,\ V_T > K] \\ &= N\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma}} N\left(\frac{\log(D^2/(KV_0)) + mT}{\sigma\sqrt{T}}\right) \end{aligned}$$

- **Payoff to Equity Investors**

$$E_T = \max(0, (V_T - K)1_{\{M_T \geq D\}})$$

European down-and-out call option with strike K, barrier $D < K$ and maturity T

$$E_0 = C(\sigma, T, K, r, V_0) - V_0 \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}+1} N(h_+) + Ke^{-rT}\left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}-1} N(h_-)$$

$$h_\pm = \frac{(r \pm \frac{1}{2}\sigma^2)T + \log(D^2/(KV_0))}{\sigma\sqrt{T}}$$

- Back-out asset price and its volatility from Equity price and its volatility

- **Payoff to Bond Investors at Maturity**

$$B_T^T = K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T < D\}}$$

Risk free loan, short European put, long down-and-in call

$$B_0^T = Ke^{-rT} - P(\sigma, T, K, r, V_0) + DIC(\sigma, T, K, D, r, V_0)$$

- **Credit Spread**

$$S(t, T) = -\frac{1}{T - t} \log \left(\frac{B_t^T}{\bar{B}_t^T} \right)$$

- Spreads $S(0, T)$ increase and then decrease with T
- Assume that total debt grows at positive rate or that fim maintains some target leverage ratio
- Time varying default barrier

KMV's Risk Neutral DP

- **Default Probability**

$$DP^{real}(t) = p(t) = 1 - P[V_t > C] = N\left(-\frac{\log(V_0/C) + (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}\right)$$

$$DP^{rn}(t) = N\left(-\frac{\log(V_0/C) + (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right)$$

$$DP^{rn}(T) = N\left(N^{-1}(DP^{real}(t)) + \frac{\mu - r}{\sigma}\sqrt{t}\right)$$

- **Expected Excess Market Return** $\pi = \mu_m - r$

- **CAPM** $\mu = r + \beta\pi,$

$$\beta = \rho_{a,m}\frac{\sigma}{\sigma_m}$$

$$DP^{rn}(t) = N\left(N^{-1}(DP^{rn}(t)) + \rho_{a,m}\frac{\pi}{\sigma_m}t^{\theta}\right)$$

- **Sharpe Ratio** π/σ_m obtained from market data

- **Spread of the Risky Zero Bond**

$$e^{-(r+s)t} = [(1 - DP^{rn}(t) + (1 - LGD)DP^{rn}(t)]e^{-rt}$$

Collateralized Debt Obligations CDO

- **CDO** is an asset backed security whose underlying collateral is typically a portfolio of corporate bonds or commercial loans
- It is a way of packaging credit risk
- Bank transfers the portfolio to an SPV and receives cash
- SPV issues fixed/floating rate notes to investors
- Notes are divided by order of seniority into tranches called equity tranche, mezzanine tranche(s) and senior tranche
- Investors receive the principal and interest from the portfolio in order of seniority

Types of CDOs

- **Cash Flow CDOs:** Collateral portfolio is not subject to active trading. All uncertainty regarding interest and principal payments is determined by timing of defaults
- **Market Value CDO:** Payments based on *mark-to-market* returns.

Economics of CDOs

CDOs become attractive due to market imperfections arising due to regulatory capital requirements, low valuations due to lack of liquidity, etc.

Two classes of CDOs

- Balance-sheet CDOs
  - Collateralized Loan Obligations (CLO) – removes loans from the banks balance sheet providing capital relief and improve valuations
  - Synthetic balance sheet CLO – No actual transfer of ownership due to client secrecy obligations or cost arising due to contractual restrictions. Use credit derivatives to transfer risk to the SPV
- Arbitrage CDOs – Capture the difference in price due to lower cost of aquiring the collateral pool in the secondary market and the value received from management fees and sale of the CDO

Tranche	% Notional	Yield	SoL	Rating
Senior	15	6	100	Aa2
Mezzanine-I	40	7.5	250	Baa3
Mezzanine-II	40	15	550	Ba1
Equity	5	35	-	NR

Table 3: CDO Example (illustrative only)

Pricing

- Model the correlated default process of individual securities comprising the portfolio
  - Copula or factor based – Static
  - Dynamic intensity models – Calibration
- Model calibrated based on market data
- Will need a model for the term structure of interest rates in case of floating notes
- Rachev *et. al.* – Calibrating the CR^+ model to reproduce iTraxx tranche quotes

Top Down Approach

- Copula based approaches found highly inadequate
- Need models to capture the dynamics in a more meaningful way
- **Top Down Approach:** Model the index on which options are to be priced
- Three Recent Papers
  - A New Framework for Dynamic Credit Portfolio Loss Modeling, *Sidenius, Piterberg, Andersen*, 2005
  - Portfolio Losses and the term structure of Loss transition rates: New methodology for the pricing of portfolio credit derivatives, *Schonbucher*, 2005
  - Pricing credit from top down with affine point processes, *Errais, Giesecke, Goldberg*, 2006

Self Affecting Point Processes

- **Hawkes Process:** $J = (L, N)$ Loss and Number of defaults share common event times that arrive with intensity

$$\lambda_t = c_t + \int_0^t d(t-s)dL_s$$

$$d(s) = \delta e^{-\kappa s}$$

- Intensity is governed by two factors
  - Timing of Past Events
  - Realized loss scaled by δ

$$c(t) = c(0)e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-s)}\rho(s)ds$$

- $\rho(s)$ level of mean reversion

$$d\lambda_t = \kappa(\rho(t) - \lambda_t)dt + \delta dL_t$$

- Hawkes arrivals are **clustered**
- At event times, J jumps according to a measure ν

Conditional Transform

- If $\rho(t)$ is constant then $\rho(t) = \lambda_\infty$

$$c(t) = \lambda_\infty + e^{-\kappa t}(c(0) - \lambda_\infty)$$

- N always has jumps of size 1. ν is the distribution of loss at default

$$E[e^{u \cdot J_s} | G_t] = e^{a(u,t,s) + b(u,t,s) + u \cdot J_t}$$

$$\begin{aligned} \partial_t b(t) &= \kappa b(t) - \phi(\delta b(t), u) + 1 \\ \partial_t a(t) &= -\kappa \lambda_\infty b(t) \end{aligned}$$

- $a(s) = b(s) = 0$

$$\phi(c, u) = e^{u \cdot (0,1)'} \int_0^\infty e^{(c + u \cdot (1,0)')z} d\nu(z)$$

- Invert transform to obtain the pdf of J_s given G_t

Index Swap

- There exists a filtered probability space with a pricing measure Q. All expectations are with respect to this measure.
- Index portfolio with n constituent name swaps with common notional I/n, common maturity date T and common premium payment dates (t_m)
- Value of Default leg at time t

$$\begin{aligned} D_t &= E\left[\int_t^T e^{-r(s-t)} dL_s | G_t\right] \\ &= e^{-r(T-t)} E[L_T | G_t] - L_t + r \int_t^T e^{-r(s-t)} E[L_s | G_t] ds \end{aligned}$$

- Premium notional I_t : Total notional on names that have survived until time t

$$I_t = I\left(1 - \frac{N_t}{n}\right)$$

- Value of premium leg

$$
\begin{aligned}
P_t(S) &= E\left[\sum_{t_m \geq t} e^{-r(t_m - t)} S \alpha_m I_{t_m} | G_t\right] \\
&= SI \sum_{t_m \geq t} e^{-r(t_m - t)} \alpha_m (1 - \frac{1}{n} E[N_{t_m} | G_t])
\end{aligned}
$$

- CDS index value at time t

$$
S_t = \frac{e^{-r(T-t)} E[L_T | G_t] - L_t + r \int_t^T e^{-r(s-t)} E[L_s | G_t] ds}{I \sum_{t_m \geq t} e^{-r(t_m - t)} \alpha_m (1 - \frac{1}{n} E[N_{t_m} | G_t])}
$$

Tranche Swap

- Lower Attachment Point: $A_L \in [0,1]$
  Upper Attachment Point: $A_U \in (A_L, 1]$
- $K = A_U - A_L$ Contract Notional I
- Tranche Notional: KI
- Cumulative payments under the Default leg

$$U_t = (L_t - IA_L)^+ - (L_t - IA_U)^+$$

- Value of Default leg payments at time $t \le T$

$$\begin{aligned} D_t &= E\left[\int_t^T e^{-r(s-t)} dU_s | G_s\right] \\ &= e^{-r(T-t)} E[U_T|G_t] - U_t + r\int_t^T e^{-r(s-t)} E[U_s|G_t] ds \end{aligned}$$

- Premium Leg has two parts:
  - Upfront payment: F fraction of the notional KI
  - Payments proportional to $I_t = KI - U_t$

$$
\begin{aligned}
P_t(F,S) &= FKI + E\left[\sum_{t_m \geq t} e^{-r(t_m - t)} S\alpha_m I_{t_m} | G_t\right] \\
&= S \sum_{t_m \geq t} e^{-r(t_m - t)} \alpha_m (KI - E[U_{t_m} | G_t])
\end{aligned}
$$

- **Premium rate** for fixed upfront payment F

$$S_t = \frac{e^{-r(T-t)}E[U_T|G_t] - U_t + r\int_t^T e^{-r(s-t)}E[U_s|G_t]ds - FKI}{\sum_{t_m \geq t} e^{-r(t_m-t)}\alpha_m(KI - E[U_{t_m}|G_t])}$$

- **Upfront payment** for fixed premium rate S

$$F = \frac{1}{KI}\Bigg(e^{-r(T-t)}E[U_T|G_t] - U_t + r\int_t^T e^{-r(s-t)}E[U_s|G_t]ds$$

$$-S\sum_{t_m \geq t} e^{-r(t_m-t)}\alpha_m(KI - E[U_{t_m}|G_t])\Bigg)$$

- **Generalization**

$$\lambda_t = \lambda(X_t, t) = \Lambda_0(t) + \Lambda_1(t) \cdot X_t$$

- $X(t)$: vector of risk factors

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t + \zeta dJ_t$$

$$d\lambda_t = \kappa(\lambda_\infty - \lambda_t)dt + \sigma dW_t + \delta dL_t$$

- **Discounted Conditional transform**

$$E\left[e^{-\int_t^s \rho(X_v,v)dv} e^{u \cdot J_u} | G_t\right] = e^{\alpha(u,t,s)+\beta(u,t,s)+u \cdot J_t}$$

Hedging Single names

- **Random Thinning**

$$N = \sum_{i=1}^{n} N^i$$

$$\lambda^i = Y^i \lambda$$

- Y_t^i Probability that name i experiences a event given that there is an event
- Choose model for Y^i process and calibrate from securities referenced on the i^{th} constituent name, say using least squares subject to non-negativity and summation constraint
- Algorithm for delta hedging

$$\Delta_t^i \approx \frac{\Delta M_t}{\Delta M_t^i}$$