MA 312 Commutative Algebra / Jan–April 2020
(BS, Int PhD, and PhD Programmes)

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Lectures : Tuesday and Thursday ; 15:30–17:00   **Venue:** MA LH-5 / LH-1

7. Rings and Modules of Fractions — Localization*

Submit a solution of ANY ONE of the *Exercise ONLY.   Due Date : Thursday, 19-03-2020
Complete Correct Solutions of the **Exercise carry BONUS POINTS !

* Localization is a very powerful technique in Commutative Algebra that often allows to reduce questions on rings and modules to union of smaller problems. It is motivated from both an algebraic and a geometric point of view.

In the following Exercises, let A be a commutative ring. For a multiplicatively closed subset $S \subseteq A$, let $\iota_S : A \to S^{-1}A$, $a \longmapsto a/1$, be the natural ring homomorphism. With this $S^{-1}A$ is endowed with the A-algebra structure with the structure homomorphism ι_S.

For an A-module V and let $\iota_S^V : V \to S^{-1}V$, $x \longmapsto x/1$, be the natural map. With the natural scalar multiplication $S^{-1}A \times S^{-1}V \to S^{-1}V$, $(a/s, x/t) \longmapsto (ax)/(st)$, the abelian group $S^{-1}V$ is endowed with the $S^{-1}A$-module structure.
Moreover, the assignment $S^{-1} : A\text{-}\mathcal{M}od \rightsquigarrow S^{-1}A\text{-}\mathcal{M}od$, $V \mapsto S^{-1}V$, defines a covariant functor from the category of A-modules to the category of S^{-1}-modules.

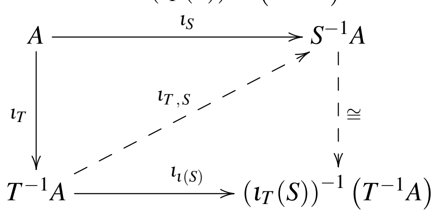
7.1 Let A be a commutative ring and let $S \subseteq A$ be a multiplicatively closed subset.

(a) Let $\mathfrak{p} \in \operatorname{Spec} A$ with $S \cap \mathfrak{p} = \emptyset$. Then the natural map $\iota : A \to S^{-1}A$ induces an isomorphism of rings $A_{\mathfrak{p}} \xrightarrow{\sim} (S^{-1}A)_{S^{-1}\mathfrak{p}}$.

(b) Let $T \subseteq A$ be a multiplicatively closed subset with $T \subseteq S$. Then the natural map $\iota_S : A \to S^{-1}A$ induces a ring homomorphism $\iota_{T,S} : T^{-1}A \to S^{-1}A$, in particular, $S^{-1}A$ is an $T^{-1}A$-algebra with the structure homomorphism $\iota_{T,S}$. Further, the $T^{-1}A$-algebra $S^{-1}A$ is canonically isomorphic to the ring of fractions of $T^{-1}A$ with respect to the image $\iota_T(S)$ of S in $T^{-1}A$ under the canonical map $\iota_T : A \to T^{-1}A$, i. e. The ring homomorphism $\iota_T : A \to T^{-1}A$ induces an $T^{-1}A$-algebra homomorphism

$$S^{-1}A \xrightarrow{\sim} (\iota_T(S))^{-1}\,(T^{-1}A)$$

such that the diagram



is commutative.

***7.2** Let A be a commutative ring. A multiplicatively closed subset S in A is called *saturated* if for all $a, b \in A$, $ab \in S$ implies that $a \in S$ and $b \in S$.

(a) For a multiplicatively closed subset $S \subseteq A$, let

$$\overline{S} := \{a \in A \mid \text{there exists } b \in A \text{ with } ab \in S\}$$

is a multiplicatively closed in A, $S \subseteq \overline{S} = \iota_S^{-1}((S^{-1}A)^{\times})$, where $\iota_S : A \to S^{-1}A$, $a \mapsto a/1$ is the natural map and $\overline{S}$ is the smallest saturated multiplicatively closed subset containing S and hence $\overline{S}$ is called the saturation of S (in A). Further, $\overline{S}$ is saturated, i. e. $\overline{(\overline{S})} = \overline{S}$ and the natural map (see Exercise 6.1 (b)) $S^{-1}A \xrightarrow{\sim} \overline{S}^{-1}A$ is an isomorphism.

(b) If $\mathfrak{p} \in \operatorname{Spec} A$, then the multiplicatively closed subset $A \smallsetminus \mathfrak{p}$ is saturated. More generally, a multiplicatively closed subset $S \subseteq A$ is saturated if and only if $A \smallsetminus S$ is a union of prime ideals.

(c) If S and T are multiplicatively closed subset in A, then the A-algebras $S^{-1}A$ and $T^{-1}A$ are isomorphic if and only if $\overline{S} = \overline{T}$.

7.3 (Total Quotient ring) Let $A \neq 0$ be a commutative ring and $\mathrm{S}_0 := \mathrm{Nzd}(A) = A \smallsetminus \mathrm{Z}(A)$ be the set of all non-zerodivisors [1] in A. Then S_0 is a multiplicatively closed subset in A. The ring of fractions $\mathrm{S}_0^{-1}A$ is called the total quotient ring of A and is usually denoted by $\mathrm{Q}(A)$. The natural ring homomorphism $\iota_{\mathrm{S}_0} : A \to \mathrm{Q}(A)$ is injective and hence A can be identified with a subring of its total quotient ring. In particular, if A is an integral domain, then $\mathrm{Q}(A)$ is the field of fractions of A (the quotient field of A).

(a) S_0 is the largest multiplicatiely closed subset of A for which the homomorphism $\iota_{\mathrm{S}_0} : A \to \mathrm{S}_0^{-1}A$ is injective.

(b) Every element in $\mathrm{Q}(A)$ is either a zerodivisor or a unit.

(c) Every non-zero ring of fractions $S^{-1}A$ of a integral domain is canonically isomorphic to a subring of the quotient field $\mathrm{Q}(A)$ of A.

(d) For every ring A in which every non-unit is a zerodivisor the natural homomorphism $\iota_{\mathrm{S}_0} : A \to \mathrm{S}_0^{-1}A$ is bijective.

7.4 Let A be an integral domain with the quotient field $K = S^{-1}A$, where $S = A \smallsetminus \{0\}$. Then in K the following equalities hold :

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \operatorname{Spm} A} A_{\mathfrak{m}} \,.$$

7.5 **(a)** Let A be an integral domain, $\mathrm{S}_0 := A \smallsetminus \{0\}$ and $K = \mathrm{S}_0^{-1}A = \mathrm{Q}(A)$ be the quotient field of A. Then $A = K$ if and only if the canonical homomorphism

$$\mathrm{S}_0^{-1}\mathrm{Hom}_A(K, A) \longrightarrow \mathrm{Hom}_{\mathrm{S}_0^{-1}(A)}\left(\mathrm{S}_0^{-1}K \,,\, \mathrm{S}_0^{-1}A\right)$$

is surjective. (**Hint :** Consider id_K ! — Once again if $K = A$ is finite over A, then $A = K$.)

(b) Let A be a commutative ring and $S \subseteq A$ be a multiplicatively closed set. If $S^{-1}A$ is a *finite* A-module, then $S^{-1}A$ is isomorphic to the A-module $A/\operatorname{Ker} \iota_S$, where $\iota : A \to S^{-1}A$ is the natural ring homomorphism.

7.6 Let A be a commutative ring.

(a) Let $S \subseteq A$ be a multiplicatively closed subset. Then S^{-1} commutes with the nilradical, i. e. $\operatorname{nil}(S^{-1}A) = S^{-1}(\operatorname{nil} A)$.

[1] **Zerodivisors in a ring** An element $a \in A$ in a ring A is called a zerodivisor in A if there exists $b \in A$, $b \neq 0$ with $ab = 0$. An element which is not a zerodivisor is called a non-zerodivisor in A. Note that 0 is a zerodivisor in A if and only if the ring $A \neq 0$. The set of all zerodivisors in the ring A is denoted by $\mathrm{Z}(A)$ and hence the $\mathrm{Nzd}(A) = A \smallsetminus \mathrm{Z}(A)$ is the set of all non-zerodivisors in A.

(b) A prime ideal $\mathfrak{p} \in (\operatorname{Spec} A, \subseteq)$ if and only if $\operatorname{Spec} A_{\mathfrak{p}}$ is singleton.

(c) If A is reduced and if $\mathfrak{p} \in (\operatorname{Spec} A, \subseteq)$ is minimal, then $A_{\mathfrak{p}}$ is a field.

(d) Let $\mathfrak{a}$ be an ideal in A and let $S_{\mathfrak{a}} := 1+\mathfrak{a} := \{1+a \mid a \in \mathfrak{a}\}$. Then S is a mutiplicatively closed set in A and $\mathfrak{a}S_{\mathfrak{a}}^{-1}A \subseteq \mathfrak{m}_{S_{\mathfrak{a}}^{-1}A}$ = the Jacobson-radical of $S_{\mathfrak{a}}^{-1}A$. What is the saturation $\overline{S_{\mathfrak{a}}}$ (see Exercise 7.2) of the multiplicatively closed set $S_{\mathfrak{a}}$?

7.7 Let A be a commutative ring, $\mathfrak{a} \subseteq A$ an ideal and let $S \subseteq A$ be a multiplicative closed subset. The residue-class homomorphism $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$ induces a canonical surjective A-algebra homomorphism $S^{-1}A \longrightarrow \pi_{\mathfrak{a}}(S)^{-1}(A/\mathfrak{a})$ with kernel $\mathfrak{a}S^{-1}A$. In particular, there is a canonical A-algebra isomorphism $S^{-1}A/\mathfrak{a}S^{-1}SA \xrightarrow{\ \sim\ } \pi_{\mathfrak{a}}(S)^{-1}(A/\mathfrak{a})$. Furthermore, there is a natural bijection

$$\operatorname{Spec} S^{-1}(A/\mathfrak{a}) \xrightarrow{\ \sim\ } \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p} \text{ and } S \cap \mathfrak{p} = \emptyset\} \subseteq \operatorname{Spec} A.$$

7.8 Let A be a commutative ring.

(a) Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal in A, $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residue field of the local ring $A_{\mathfrak{p}}$, $\mathrm{Q}(A/\mathfrak{p})$ be the field of fractions of the integral domain $A/\mathfrak{p}$ and $\pi_{\mathfrak{p}} : A \to A/\mathfrak{p}$, $\pi_{\mathfrak{p}A_{\mathfrak{p}}} : A_{\mathfrak{p}} \to \kappa(\mathfrak{p})$ be the canonical residue-class homomorphisms. Then there exists a natural isomorphism $\sigma_{\mathfrak{p}} : \mathrm{Q}(A/\mathfrak{p}) \longrightarrow \kappa(\mathfrak{p})$ $(\pi_{\mathfrak{p}}(a)/\pi_{\mathfrak{p}}(s) \longmapsto \pi_{\mathfrak{p}A_{\mathfrak{p}}}(a/s)$ of fields such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\ \iota\ } & A_{\mathfrak{p}} & \xrightarrow{\ \pi_{\mathfrak{p}A_{\mathfrak{p}}}\ } & \kappa(\mathfrak{p}) \\ \| & & & & \cong \uparrow \sigma_{\mathfrak{p}} \\ A & \xrightarrow{\ \pi_{\mathfrak{p}}\ } & A/\mathfrak{p} & \xrightarrow{\ \iota\ } & \mathrm{Q}(A/\mathfrak{p}) \end{array}$$

is commutative. We shall use $\sigma_{\mathfrak{p}}$ to identify $\kappa(\mathfrak{p})$ and $\mathrm{Q}(A/\mathfrak{p})$. With this for $f \in A$, the image of f under either composite $\pi_{\mathfrak{p}A_{\mathfrak{p}}} \circ \iota$ or $\iota \circ \pi_{\mathfrak{p}}$ is denoted by $f(\mathfrak{p})$ and is called the v a l u e of f at $\mathfrak{p}$.

(b) The ring A is reduced if and only if the map

$$A \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Spec} A} \kappa(\mathfrak{p}),\ f \longmapsto (f(\mathfrak{p}))_{\mathfrak{p} \in \operatorname{Spec} A}$$

is injective. (**Remark :** This means for a reduced ring A, an element $f \in A$ is zero if and only if it is the zero function on $\operatorname{Spec} A$.)

7.9 Let A be a commutative ring and let $T = \{t_i \mid i \in I\}$ be a family of elements in A and let $S := \langle T, \cdot \rangle \subseteq A$ be the multiplicative submonoid of $(A, \cdot)$ generated by T, i. e. S consists of all finite products of elements in T. Then there exists a canonical isomorphism of A-algebras

$$S^{-1}A \xrightarrow{\ \sim\ } A[X_i \mid i \in I]/\langle t_i X_i - 1 \mid i \in I \rangle.$$

In particular, if T is finite, then $S^{-1}A$ is a finite type algebra over A generated by $S^{-1} := \{1/s \mid s \in S\}$. If $T = \{t\}$, then $A_t \xrightarrow{\sim} A[X]/\langle tX-1 \rangle$ is a cyclic A-algebra.

7.10 The localization $A[X]_X = S^{-1}(A[X])$ of the polynomial ring over a ring A with $S = \{X^n \mid n \in \mathbb{N}\}$ is the so-called ring of L a u r e n t p o l y n o m i a l s over A usually denoted by $A[X, X^{-1}]$ which consists of all formal expressions of type $\sum_{n \in \mathbb{Z}} a_n X^n$, where $(a_n)_{n \in \mathbb{Z}} \in A^{(\mathbb{Z})}$ endowed with conventional addition and multiplication.

7.11 Let $A[X_i \mid i \in I]$ be the polynomial rings over a ring A in indeterminates X_i, $i \in I$. The for a multiplicatively closed subset $S \subseteq A$, there exists a canonical isomorphism of rings

$$S^{-1}(A[X_i \mid i \in I]) \xrightarrow{\sim} \left(S^{-1}A\right)[X_i \mid i \in I].$$

7.12 Let $K[X,Y]$ be the polynomial ring over a field K, $\mathfrak{a} = \langle X^2, XY\rangle \subseteq K[X,Y]$ and $S = K[X,Y] \smallsetminus \langle X\rangle$. Then (in contrast to the case when $\mathfrak{a} = \mathfrak{p} \in \operatorname{Spec} K[X,Y]$)

$$\mathfrak{a} \subsetneq \mathfrak{a}S^{-1}(K[X,Y]) \cap K[X,Y].$$

7.13 Let A be a integral domain.

(a) (Lemma of Nagata) *Suppose that every $a \in A \smallsetminus (\{0\} \cup A^\times)$ has a irreducible factorisation,* i. e. *is a product of* irreducible *elements in A (for example, A is a noetherian integral domain) and that $S \subseteq A$ is a multiplicatively closed subset with $0 \notin S$ and every non-unit in S has a prime factorisation,* i. e. *is product of* prime *elements in A. Then if $S^{-1}A$ is a factorial domain, then A is also a factorial domain.*

(b) (Theorem of Gauss) *If A is a factorial domain, then the polynomial ring $A[X]$ is also a factorial domain.* (**Hint :** We give a proof using the Lemma of Nagata in Part (a). Let $S := A \smallsetminus \{0\}$. Then $S^{-1}(A[X]) \xrightarrow{\sim} \left(S^{-1}A\right)[X]$ (see Exercise 7.11) is a PID, since $S^{-1}A = \mathrm{Q}(A)$ is the quotient field of A and hence a factorial domain. Further, since A is factorial, S is generated by prime elements in A which are also prime elements in $A[X]$ (proof?). Now use Lemma of Nagata to conclude that $A[X]$ is factorial. — **Remark :** Lemma of Nagata is very useful to produce many examples of factorial domains. For example, one can use it to prove the following theorem :

(Klein-Nagata) *Let K be a field of Characteristic $\neq 2$. For every natural number $n \geq 5$ and arbitrary non-zero elements $a_1, \ldots, a_n \in K$, the finite type K-algebra*

$$A := K[X_1, \ldots, X_n]/\langle a_1X_1^2 + \cdots + a_nX_n^2 \rangle$$

is a factorial integral domain.)

7.14 Let A be a commutative ring, $S \subseteq A$ be a multiplicatively closed set in A and let V be an A-module. We say that an element $a \in A$ is a non-zerodivisor on V if the map $\lambda_a : V \to V$, $x \mapsto ax$ is injective. If $a \in A$ is a non-zerodivisor on V, then $a/1$ is a non-zerodivisor on the $S^{-1}A$-module $S^{-1}V$. In particular, if $a \in A$ is a non-zerodivisor in A, then $a/1$ is a non-zerodivisor in $S^{-1}A$.

7.15 Let V be an A-module, $S \subseteq A$ be a multiplicatively closed subset and let $\iota_S^V : V \to S^{-1}V$, $x \longmapsto x/1$, be the natural map.

(a) $\operatorname{Ker} \iota_S^V = \{x \in V \mid sx = 0 \text{ for some } s \in S\}$. In particular, ι_S^V is injective if and only if $\lambda_s : V \to V$, $v \mapsto sv$, is injective for every $s \in S$. If $\mathrm{S}_0 = \operatorname{Nzd}(A)$ is the set of all non-zerodivisors in A, then $\operatorname{Ker} \iota_{\mathrm{S}_0}^V = \mathrm{t}_A V = \{x \in V \mid sx = 0 \text{ for some } s \in \mathrm{S}_0\}$ is the torsion-submodule of V.

(b) The map ι_S^V is bijective if and only if all λ_s, $s \in S$, are bijective. In this case, there is a unique $S^{-1}A$-module structure on V which is induced by the given A-mdule structure on V. the scalar multiplication of $S^{-1}A$ on V is : $(a/s) \cdot x = a\lambda_s^{-1}(x)$, $a \in A$, $x \in V$.

(c) The natural ring homomorphism $\iota_S : A \to S^{-1}A$ is bijective if and only if $S \subseteq A^\times$.

7.16 For A-submodules U and U' of an A-module V and for ideals $\mathfrak{a}$, $\mathfrak{b}$ in A, we have

(1) $S^{-1}(U \cap U') = S^{-1}U \cap S^{-1}U'$. **(2)** $S^{-1}(U + U') = S^{-1}U + S^{-1}U'$.

(3) $S^{-1}(U : U') = \left(S^{-1}U : S^{-1}U'\right)$ if the submodule U' is finitely generated

(4) $S^{-1}(\mathfrak{a}U) = S^{-1}(\mathfrak{a})S^{-1}U$. **(5)** $S^{-1}(\mathfrak{a}\mathfrak{b}) = S^{-1}\mathfrak{a}S^{-1}\mathfrak{b}$.

(6) $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})$. **(7)** $S^{-1}(\sqrt{\mathfrak{a}}) = \sqrt{S^{-1}\mathfrak{a}}$.

(8) $S^{-1}(\mathfrak{a}:\mathfrak{b}) = \big(S^{-1}\mathfrak{a}:S^{-1}\mathfrak{b}\big)$ if the ideal $\mathfrak{b}$ is finitely generated.

7.17 Let A be a commutative ring and V be a *finite* A-module. For a multiplicatively closed subset S in A, show that $S^{-1}V=0$ if and only if $sV=0$ for some $s\in S$, i. e. $S\cap \operatorname{Ann}_A V\neq 0$.

7.18 (Lemma of Dedekind) Let A be a commutative ring, V be a finite A-module and $\mathfrak{a}$ be an ideal in A with $V=\mathfrak{a}V$. Show that $(1+a)V=0$ for some $a\in\mathfrak{a}$. (**Hint:** Note that $(1+\mathfrak{a})^{-1}V_{1+\mathfrak{a}}=0$ by Exercise 6.6 (d) and the Lemma of Krull-Nakayama[2] — **Another elementary proof:** Suppose that $V=Ax_1+\cdots+Ax_n$ and $V_i:=Ax_1+\cdots+Ax_i$, $i=0,\ldots,n$. By induction show that there are elements $a_j\in\mathfrak{a}$ such that $(1-a_j)V\subseteq\mathfrak{a}V_{n-j}$, $j=0,\ldots,n$.)

7.19 (Modules with rank) Let A be a non-zero commutative ring, S_0 be the multiplicatively closed subset of non-zerodivisors in A and $\mathrm{Q}(A)=\mathrm{S}_0^{-1}A$ be the total quotient ring of A (see Exercise 6.3). An A-module V is called a module with rank over A if $\mathrm{S}_0^{-1}V$ is a free $\mathrm{Q}(A)$-module; in this case, we also say that V has rank over A and put $\operatorname{Rank}_A V:=\operatorname{Rank}_{\mathrm{Q}(A)}\mathrm{S}_0^{-1}V$.

(a) Every free A-module V is n A-module with rank and in this case its rank is nothing but the rank of the free A-module V, i. e. *the* cardinality of an A-basis of V.

(b) If A is an integral domain, then $\mathrm{Q}(A)$ is the quotient field of A and hence every A-module V has rank and $\operatorname{Rank}_A V=\operatorname{Dim}_{\mathrm{Q}(A)}\mathrm{S}_0^{-1}V$.

(c) If V is an A-module with rank, then $\mathrm{S}_0^{-1}V$ has a $\mathrm{Q}(A)$-basis of the type $x_i/1$, $i\in I$ and x_i, $i\in I$, s a maximal linearly independent (over A) family in V.

(d) Every finite torsion-free A-module with rank is isomorphic to a A-submodule of a finite free A-module.

7.20 Let A be a commutative ring and let V be a projective A-module (i. e. V is a direct summand of a free A-module). Let S_0 be the multiplicatively closed subset of non-zerodivisors in A. If $\mathrm{S}_0^{-1}V$ is a finite $\mathrm{Q}(A)=\mathrm{S}_0^{-1}A$-module, then V is a finite A-module. — In particular, a *projective module*[3] over an integral domain is finite if and only if it has a finite rank. (**Hint:** Let f be an embedding of V as a direct summand in a free A-module of the type $A^{(I)}$, I an indexed set and consider the image of $S_0^{-1}f$.)

7.21 If V is a noetherian (resp. artinian) A-module over a commutative ring, then $S^{-1}V$ is a noetherian (resp. artinian) $S^{-1}A$-module

*__7.22__ Let A be a commutative ring, S a multiplicatively closed subset in A and V, W be modules over A. For the canonical homomorphism

$$\Phi_V : S^{-1}\operatorname{Hom}_A(V,W)\longrightarrow \operatorname{Hom}_{S^{-1}A}\big(S^{-1}V\,,S^{-1}W\big)\ ,\ \ f/s\longmapsto (x/s\mapsto f(x)/s)$$

the following assertions hold:

(a) If V is a finite A-module, then Φ_V injective.

[2] **Lemma of Krull-Nakayama** *Let A be a commutative ring, $\mathfrak{a}$ be an ideal in A. The following statements are equivalent:* (i) $\mathfrak{a}\subseteq\mathfrak{m}_A$. (ii) *For every A-module V and every submodule U of V with V/U finitely generated, the following implication hold: If $V=U+\mathfrak{a}V$, then $V=U$.*

[3] Recall that an A-module P is called projective over A if it is isomorphic to direct summand of a free A-module. Equivalently, every short exact sequence $0\to V'\xrightarrow{f'}V\xrightarrow{f}P\to 0$ of A-modules *splits*. See Footnote No. 1 in Exercise Set 05.

(b) If V is a finite A-module and if the canonical homomorphism $W \to S^{-1}W$ injective (in this case one say that W is S-torsion-free), then Φ_V bijective.

(c) If V is *finitely presented* (see Exercise Set 05 Footnote No. 3) A-module, then Φ_V bijective. (**Hint:** For a proof of (c), first note that:

For any indexed set I and any A-module W, the natural map $\operatorname{Hom}_A(A^{(I)}, W) \xrightarrow{\sim} W^i$, $f \mapsto (f(e_i))_{i\in I}$ *is an isomorphism of A-modules, where* e_i, $i \in I$, *is the standard basis of the free A-module* $A^{(I)}$.

Now, consider an exact sequence $G \xrightarrow{f} F \xrightarrow{g} V \longrightarrow 0$ with finite free A-modules F, G and the canonical commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & S^{-1}\operatorname{Hom}_A(V,W) & \xrightarrow{g'} & S^{-1}\operatorname{Hom}_A(F,W) & \xrightarrow{g} & S^{-1}\operatorname{Hom}_A(G,W) \\
 & \downarrow{\scriptstyle \Phi_V} & & \downarrow{\scriptstyle \Phi_F} & & \downarrow{\scriptstyle \Phi_G} \\
0 \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}V,S^{-1}W) & \xrightarrow{f'} & \operatorname{Hom}_{S^{-1}A}(S^{-1}F,S^{-1}W) & \xrightarrow{f} & \operatorname{Hom}_{S^{-1}A}(S^{-1}G,S^{-1}W)
\end{array}$$

with exact rows, Φ_F, Φ_G are bijective and hence Φ_V is bijective.)

7.23 Let K be a field, I be an *infinite* indexed set and $A := K^I$, $\mathfrak{a} := K^{(I)}$ ideal in A and let $S := \{(s_i)_{i\in I} \in K^I \mid s_i \neq 0 \text{ for almost all } i \in I\}$. Then S is a multiplicatively closed in A.

(a) The canonical homomorphism

$$\Phi_{A/\mathfrak{a}} : S^{-1}\operatorname{Hom}_A(A/\mathfrak{a},A) \longrightarrow \operatorname{Hom}_{S^{-1}A}((S^{-1}(A/\mathfrak{a}), S^{-1}A)$$

is *not* surjective. (**Hint:** The map $f \mapsto f(1_{A/\mathfrak{a}})$ shows $\operatorname{Hom}_A(A/\mathfrak{a},A) \cong \operatorname{Ann}_A \mathfrak{a} = 0$ and $S^{-1}\mathfrak{a} = 0$.)

(b) For every *infinite* set J, the canonical homomorphism

$$\Phi_{A^{(J)}} : S^{-1}\left(\operatorname{Hom}_A(A^{(J)}, A)\right) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}A^{(J)}, S^{-1}A)$$

is *not* injective.

Local-global Principle:

A *local-global principle* is a theorem that states that some prperty holds "globally" if and only it holds everywhere "locally."

7.24 Let V be an A-module over a commutative ring A.

(a) Then the following statements are equivalent:

(i) $V = 0$. (ii) $V_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$. (iii) $V_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \operatorname{Spm} A$.

(b) Let U, U' be A-submodules of V. Then $U = U'$ if and only if $U_\mathfrak{m} = U'_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Spm} A$.

(c) Let $x, y \in V'$. Then $x = y'$ if and only if $x/1 = y/1$ in $V_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Spm} A$.

7.25 Let V be an A-module over a commutative ring A and let $\mathfrak{a} \subseteq A$ be an ideal. Suppose that $V_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \operatorname{Spm} A$ with $\mathfrak{m} \supseteq \mathfrak{a}$. Then $V = \mathfrak{a}V$. (\bf Hint: Pass to the $A/\mathfrak{a}$-module $V/\mathfrak{a}V$ and use Exercise 7.24.)

7.26 Let A be a ring.

(a) *Being reduced is a local property* i. e. a ring A is reduced if and only if $A_\mathfrak{m}$ is reduced for all $\mathfrak{m} \in \operatorname{Spm} A$.

(b) *Being an integral domain is not local property*, i. e. a ring A might not be an integral domain although the localizations $A_\mathfrak{m}$ are integral domains for all $\mathfrak{m} \in \operatorname{Spm} A$.

7.27 Let A be an integral domain, $S \subseteq A$ be a multiplicatively closed subset in A and let V be an A-module. Then $\mathrm{t}_{S^{-1}A} S^{-1}V = S^{-1}\mathrm{t}_A V$ (recall that $\mathrm{t}_A V$ denote the torsion-submodule of V). Deduce that the following statements are equivalent:

(i) V is torsion-free.

(ii) $V_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$.

(iii) $V_{\mathfrak{m}}$ is torsion-free for all prime ideals $\mathfrak{m} \in \operatorname{Spm} A$.

7.28 Let K be a field, I an *infinite* indexed set and A be the product ring K^I. For every $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is a field. In particular, $\mathfrak{p} \in \operatorname{Spm} A$.

7.29 Let A be a commutative *semi-local* ring, i. e. the maximal spectrum $\operatorname{Spm} A$ is finite. An A-module V is free of rank r if and only if $V_{\mathfrak{m}}$ is free of rank r over $A_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{Spm} A$. (**Hint:** One can compute modulo the Jacobson-radical $\mathfrak{m}_A = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ of A and note that $A/\mathfrak{m}_A$ is the product of fields $A/\mathfrak{m}_i$, $i = 1, \ldots, n$.)

7.30 Let A be a commutative ring.

(a) A sequence $V' \to V \to V''$ of A-modules is exact if and only if for every $\mathfrak{m} \in \operatorname{Spm} A$, the sequence $V'_{\mathfrak{m}} \to V_{\mathfrak{m}} \to V''_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$-modules is exact.

(b) An A-module homomorphism $f : V \to W$ is injective (resp. surjective, resp. bijective, resp. zero) if and only if $f_{\mathfrak{m}} : V_{\mathfrak{m}} \to W_{\mathfrak{m}}$ is injective (resp. surjective, resp. bijective, resp. zero) for every maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$, i. e. *being injective, surjective, bijective and zero-ness of a module homomorphisms are local properties.*

(c) Let U be an A-submodule of an A-module V and $x \in V$. Then $x \in U$ if and only if $x/1 \in U_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{Spm} A$, i. e. *being an element of a submodule is a local property.*

7.31 Let V be an A-module of finite presentation over a commutative ring A. Then V is a projective A-module if and only if for all $\mathfrak{m} \in \operatorname{Spm} A$ the localizations $V_{\mathfrak{m}}$ are projective $A_{\mathfrak{m}}$-modules. (**Remark:** In general, being projective module is not a local property. But the projective modules are always locally free.)

*__7.32__ Let A be a commutative ring and let $B = A[x]$ be a finite cyclic (commutative) free A-algebra of rank $n \in \mathbb{N}$. Then there exists a unique monic polynomial $f \in A[X]$ of degree n which generates the kernel of the substitution A-algebra homomorphism $\varepsilon_x : A[X] \to B$, $X \mapsto x$. oreover, if $\mathfrak{m} \in \operatorname{Spm} A$ is a maximal ideal in A and if $\bar{x}$ denote the residue-class of x in $B/\mathfrak{m}B$, then the residue-class of f in $(A/\mathfrak{m})[X]$ is the minimal monic polynomial $\mu_{\bar{x}, A/\mathfrak{m}}$. (**Hint:** To show that B is a free A-module with basis $1, x, \ldots, x^{n-1}$, consider the A-module homomorphism $g : A^n \to B$, $e_i \mapsto x^{i-1}$, $i = 1, \ldots, n$ where e_i, $i = 1, \ldots n$, is the standard A-basis of the free A-module A^n. Now, use Exercise 7.30 (b) to conclude that g is an isomorphism of A-modules.)

**__7.33__ (Kronecker Extensions) For a system U_i, $i \in I$, of indeterminates over a (commutative) ring A, we use the short notation $A[U] := A[U_i \mid i \in I]$. For a polynomial $F \in A[U]$, the ideal $\mathrm{C}(F)$ generated by the coefficients of F in A is called the content of F. A polynomial $F \in A[U]$ is called a primitive if its content $\mathrm{C}(F)$ is a unit ideal.

(a) A polynomial $F \in A[U]$ is primitive if and only if for every maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$, the residue-class of F in $(A/\mathfrak{m})[U]$ is not the zero-polynomial.

(b) The set $S \subseteq A[U]$ of all primitive polynomials in $A[U]$ is a saturated multiplicatively closed subset in $A[U]$. (**Hint:** Use the Lemma of McCoy which states that: *If $F \in A[U]$ is non-zerodivisor, then there exists an element $a \in A$, $a \neq 0$, with $aF = 0$.*)

(c) For arbitrary family U_i, $i \in I$, of indeterminates the A-algebra

$$A(U) = A(U \mid i \in I) := S^{-1}A[[U_i \mid i \in I],$$

where S is the multiplicatively closed subset of all primitive polynomials in $A[U]$ is called the Kronecker extension[4] of A (in the indeterminates U_i, $i \in I$.)
Every Kronecker extension $A \to A(U)$ is faithfully flat. In particular, $A(U)^\times \cap A = A^\times$. Moreover, the canonical map $\operatorname{Spec} A(U) \longrightarrow \operatorname{Spec} A$ induces a homeomorphism (with respect to the Zariski topologies) $\operatorname{Spm} A(U) \xrightarrow{\sim} \operatorname{Spm} A$. Every maximal ideal of $A(U)$ is the extension of a maximal ideal of A.
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[4] Kronecker extensions provide the conceptual tools for Kronecker's *method of indeterminates* ("Unbestimmten-Methode'). Special cases of it have been used for a long time, for instance, see Exercise 4.27. A modern use of the method can be found in Nagata's book [Nagata, M. :, Local rings, Intersc. Publ.,New York 1962].

[5] **Faithfully flat algebras** Let A be a commutative ring and let B is an A-algebra with the structure homomorphism $\varphi : A \to B$. We say that B is faithfully flat A-algebra if B is a flat A-module.

(a) If $B \neq 0$ is free A-algebra, then B is faithfully flat over A.

(b) If B is flat over A, then the following statements are equivalent :

(i) B is a faithfully flat A-algebra. (ii) B is a pure A-algebra.

(iii) For every ideal $\mathfrak{a}$ in A, $\varphi^{-1}B\mathfrak{a}) = \mathfrak{a}$. (iv) For every maximal ideal $\mathfrak{m} \in \operatorname{Spm} A$ in A, $B\mathfrak{m} \neq B$.

(c) Let $A \subseteq B$ be a flat extension of commutative rings. If B is integral over A, then B is faithfully flat over A.

(**Hint :** For a proof of the implication (i)⇒(ii) : Let V be an arbitrary A-module, $\iota : V \to V_{(B)} = B \otimes V$ the canonical map. Then there exists a B-module homomorphism $h : (V_{(B)})_{(B)} \longrightarrow V_{(B)}$ with $h \circ \iota_{(B)} = \mathrm{id}$. It follows that $\iota_{(B)}$ is injective and hence ι is also injective.)