E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Nikhil Gupta** (nikhil.gupta@csa.iisc.ernet.in ; **Lab No.: 303**) / **Vineet Nair** (vineetn90@gmail.com ; **Lab No.: 303**) / **Rahul Gupta** (rahul.gupta@csa.iisc.ernet.in ; **Lab No.: 224**) / **Sayantan Mukherjee** (meghanamande@gmail.com ; **Lab No.: 253**) / **Palash Dey** (palash@csa.iisc.ernet.in ; **Lab No.: 301 , 333 , 335**)

Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

5. Linear Maps

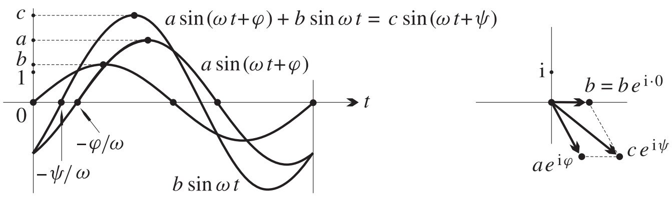
Submit a solution of the $*$-Exercise ONLY. **Due Date :** Wednesday, 07-09-2016 (Before the Class)

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

5.1 (Pointer representation) Let $\omega \in \mathbb{R}_+^\times$ and W be the $\mathbb{R}$-vector space of the functions $a\sin(\omega t + \varphi)$, $a, \varphi \in \mathbb{R}$, with basis $\sin \omega t$, $\cos \omega t$,(see Exercise 4.1). Then the map

$$\gamma : a\sin(\omega t + \varphi) \longmapsto ae^{i\varphi}, a \geq 0,$$

is a $\mathbb{R}$-vector space isomorphism of W onto $\mathbb{C}$. (**Remark :** This isomorphism is called the pointer representation of the simple harmonic motion with the circular frequency ω. The differentiation in W correspond to the multiplication by $i\omega$ to the pointer representation, i.e. $\gamma(\dot{x}) = i\omega\,\gamma(x)$ for $x \in W$. In the representation $ae^{i\varphi}$ of $a\sin(\omega t + \varphi)$, $a \geq 0$, $a = |ae^{i\varphi}|$ is called the (maximal) amplitude and $e^{i\varphi}$ is called the phase factor.)



***5.2** Let $I \subseteq \mathbb{R}$ be an interval with more than one point and let $a \in I$. For $n \in \mathbb{N}^*$, let

$$T_{a,n} : \mathrm{C}_{\mathbb{K}}^{n-1}(I) \to \mathbb{K}[t]_n$$

be the map which maps every function $f \in \mathrm{C}_{\mathbb{K}}^{n-1}(I)$ to its *Taylor-polynomial* of degree $< n$ of f at a, i. e.,

$$f \mapsto T_{a,n}(f) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k .$$

Show that $T_{a,n}$ is $\mathbb{K}$-linear. Determine the kernel and the image of this map $T_{a,n}$. (**Remark :** See also Exercise 3.5 and Supplement S5.15.)

5.3 Let V be a K-vector space with $\mathrm{Dim}_K V \geq 2$ (i. e. V contain at least two linearly independent vectors). Then every additive map $f : V \to V$ with $f(Kx) \subseteq Kx$ for all $x \in V$ is a homothecy $\vartheta_a : V \to V$, $x \mapsto ax$, of V by a scalar $a \in K$.

5.4 Let $f_1 : V \to V_1$ and $f_2 : V \to V_2$ be homomorphisms of K-vector spaces. The K-linear map $f : V \to V_1 \times V_2$ defined by $f(x) = \big(f_1(x), f_2(x)\big)$ is an isomorphism if and only if f_1 surjective and the restriction $f_2|_{\mathrm{Ker} f_1} : \mathrm{Ker} f_1 \to V_2$ is bijective.

†**5.5** (Characters) Let M and N be two monoids with neutral elements e_M and e_N, respectively. A map $\varphi : M \to N$ is called a (monoid-)homomorphism if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in M$ and $\varphi(e_M) = e_N$.

Let M be a monoid and let K be a field. By a character of M in K we mean a homomorphism of M in the multiplicative group $(K^\times, \cdot)$ of K. The map $M \to K^\times$, $x \mapsto 1_K$ is a character of M in K, called the trivial character. If $a \in K^\times$, then the conjugation by a $\varkappa_a : K \to K^\times$, $b \mapsto aba^{-1}$ is a character of the multiplicative monoid of K with values in K.

(Lemma of Dedekind-Artin[1]) Let M be a monoid and et K be a field. Then the set $\chi(M,K)$ of characters of M in K is linearly independent (in the K-vector space K^M of all K-valued functions on M) over K. (**Hint:** Suppose that $a_1\chi_1 + \cdots + a_n\chi_n = 0$ with $a_1, \ldots, a_n \in K^\times$, pairwise distinct $\chi_1, \ldots, \chi_n \in \chi(M,K)$ is a linear dependence relation with minimal $n \in \mathbb{N}$. Note that $n \geq 2$, since every character $\chi \neq 0$. Let $x \in M$ be fixed and $y \in M$ be arbitrary. Then $0 = (a_1\chi_1 + \cdots + a_n\chi_n)(xy) = a_1\chi_1(xy) + \cdots + a_n\chi_n(xy) = a_1\chi_1(x)\,\chi_1(y) + \cdots + a_r\chi_n(x)\,\chi_r(y)$ and hence $a_1\chi_1(x)\chi_1 + \cdots + a_n\chi_n(x)\chi_n = 0$. Now, conclude that $\chi_1 = \cdots = \chi_n$ a contradiction to $n \geq 2$.)

[1] This assertion is used frequently (especially in *Galois Theory*).