E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

Supplement 4

Dimension of vector spaces

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

S4.1 Compute the dimension of $U, W, U+W$ and $U \cap W$ for the following subspaces U, W of the given vector space V.

(a) $V := \mathbb{R}^3$, $U := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1+x_2=0,\ -x_2+x_3=0\}$,
$W := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1+x_3=0,\ x_1-x_2-x_3=0\}$.

(b) $V := \mathbb{R}^4$, $U := \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1-x_2+x_3=0,\ x_1+x_2-x_4=0\}$,
$W := \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1+x_2-3x_3=0,\ x_1+2x_3-x_4=0\}$.

(c) $V := \mathbb{R}^5$, $U := \mathbb{R}x_1+\mathbb{R}x_2+\mathbb{R}x_3$, $W := \mathbb{R}y_1+\mathbb{R}y_2$ mit $x_1 := (1,1,0,1,0)$, $x_2 := (0,1,1,0,1)$, $x_3 := (0,1,1,0,0)$, $y_1 := (0,0,1,1,0)$, $y_2 := (1,1,-1,0,-1)$.

S4.2 Let $n \in \mathbb{N}$, $n \geq 2$. Determine whether or not the vectors

(a) $(1,1,\ldots,1), (1,2,1,\ldots,1), \ldots, (1,\ldots,1,n)$ form a basis of $\mathbb{R}^n$ (resp. $\mathbb{Q}^n$).

(b) $(-(n-1),1,\ldots,1), (1,-(n-1),1,\ldots,1), \ldots, (1,\ldots,1,-(n-1))$ form a basis of $\mathbb{R}^n$ (resp. $\mathbb{Q}^n$).

S4.3 **(a)** Let $W \subseteq \mathbb{R}^4$ be the subspace generated by $y_1 := (1,2,3,4)$, $y_2 := (4,3,2,1)$, $y_3 := (-1,0,1,2)$, $y_4 := (0,1,0,1)$, $y_5 := (1,3,-2,0)$. List all bases of W which are the subsequences of $y_1, \ldots, y_5$.

(b) Let $U \subseteq \mathbb{R}^4$ be the subspace generated by the vectors $x_1 := (0,12,-3,10)$, $x_2 := (1,7,-3,2)$, $x_3 := (-1,5,0,7)$, $x_4 := (1,3,-2,-1)$ and let $W \subseteq \mathbb{R}^4$ be the subspace as in the part (a).

(1) From $x_1, \ldots, x_4$ choose a basis of U and extend it to a basis of $U+W$ by using the vectors $y_1, \ldots, y_5$. (2) Find a basis of $U \cap W$.

S4.4 Compute the co-ordinates of the vectors

(a) $(\mathrm{i},0), (1+\mathrm{i},-2+3\mathrm{i}), (0,1)$ with respect to the basis $v_1 = (1+\mathrm{i},\mathrm{i}), v_2 = (1,1+\mathrm{i})$ of the $\mathbb{C}$-vector space $\mathbb{C}^2$.

(b) $(1,0,-5\mathrm{i}), (2+\mathrm{i},1,0)$ with respect to the basis $v_1 = (1,0,1-\mathrm{i}), v_2 = (2+\mathrm{i},-1,-\mathrm{i}), v_3 = (0,1+\mathrm{i},2-\mathrm{i})$ of the $\mathbb{C}$-vector space $\mathbb{C}^3$.

S4.5 Let K be a field. For which $(a,b)\in K^2$, the vectors $(a,b),(b,a)$ for a basis of K^2.

S4.6 Show that the elements $x_1,\ldots,x_n$ of the K-vector space V are linearly independent if and only if the subspace $U := Kx_1+\cdots+Kx_n$ has dimension n.

S4.7 Let x_i, $i\in I$, be a family of vectors in a K-vector space V and let U be a subspace of V generated by x_i, $i\in I$. Show that U is finite dimensional if and only if there exists a natural number $n\in\mathbb{N}$ such that every $n+1$ vectors among x_i, $i\in I$, are linearly dependent. Moreover, if this condition is satisfied then the dimension $\mathrm{Dim}_K U$ is the minimum of the $n\in\mathbb{N}$ with this property.

S4.8 Let K be a finite field with q elements. Show that a K-vector space of dimension $n\in\mathbb{N}$ has exactly q^n elements.

S4.9 Let K be a finite field with q elements.

(a) The multiples $m\cdot 1_K$, $m\in\mathbb{Z}$, form a subfield K' of K.

(b) There exists a smallest positive natural number p such that $p\cdot 1_K=0$. Moreover, it is prime (and is called the C h a r a c t e r i s t i c of K — denoted by $\mathrm{Char}\,K$). The subfield $K'\subseteq K$ contains exactly p distinct elements $0, 1_K,\ldots,(p-1)1_K$.

(c) Show that $q=p^n$ with $n:=\mathrm{Dim}_{K'}K$.

(**Remark:** *The number of elements is a finite field is a power of a prime number.* Conversely, for a given prime-power q there exists (essentially unique) field with q elements, for a proof see ???.)

S4.10 Let V be a finite dimensional K-vector space and let U be a subspace of V. Let $u_1,\ldots,u_m$ be a basis of U and let $u_1,\ldots,u_m,u_{m+1},\ldots,u_n$ be an extended basis of V. Show that

$$x=a_1u_1+\cdots+a_mu_m+b_{m+1}u_{m+1}+\cdots+b_nu_n\in V$$

is an element of U if and only if the coordinates $b_{m+1}=u^*_{m+1}(x),\ldots,b_n=u^*_n(x)$ of x with respect to the basis $u_1,\ldots,u_n$ of V are zero. (**Remark:** This is the most common method of characterizing the elements of a subspace.)

S4.11 Let V be a $\mathbb{C}$-vector space of dimension $n\in\mathbb{N}^*$ and let H be a real hyperplane in V (i. e. a real subspace of dimension $2n-1$). Then show that $H\cap \mathrm{i}H$ is a complex hyper-plane in V (i. e. a complex subspace of dimension $n-1$), where we put $\mathrm{i}H:=\{\mathrm{i}x\mid x\in H\}$.

S4.12 Let U_1, U_2, U_3 be finite dimensional subspaces of a K-vector space V with $U_i\cap U_j=0$ for $i\neq j$. Show that

$$\begin{aligned}\mathrm{Dim}\,\big((U_1+U_2)\cap U_3\big) &= \mathrm{Dim}\,\big((U_1\cap(U_2+U_3)\big)\\ &= \mathrm{Dim}\,U_1+\mathrm{Dim}\,U_2+\mathrm{Dim}\,U_3-\mathrm{Dim}\,(U_1+U_2+U_3)\,.\end{aligned}$$

S4.13 Let V be a K-vector space with a countably infinite basis. Show that for every subspace U of V there exists a countable basis. (**Hint:** Let x_i, $i\in\mathbb{N}$, be a basis of V and let $V_n:=Kx_0+\cdots+Kx_n$. Then $U=\bigcup_{n=0}^{\infty}(U\cap V_n)$.)

S4.14 Let U be the subspace generated by the following functions in a space of a;; real-valued functions on $\mathbb{R}$. Compute the dimension of U, by choosing a basis from the given generating system and expressing other functions in this generating system as the linear combinations of the basis chosen.

(a) t^2, $(t+1)^2$, $(t+2)^2$, $(t+3)^2$. **(b)** $\sinh 3t$, $\cosh 3t$, e^{3t}, e^{-3t}.

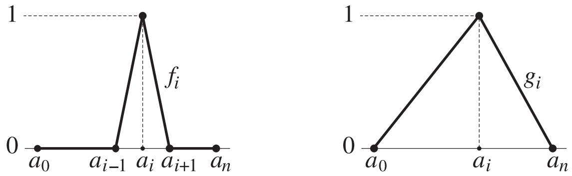
(c) 1, $\sin t$, $\sin 2t$, $\sin^2 t$, $\cos t$, $\cos 2t$, $\cos^2 t$. **(d)** 1, $\sinh t$, $\sinh 2t$, $\sinh^2 t$, $\cosh t$, $\cosh 2t$, $\cosh^2 t$.

S4.15 Let $n\in\mathbb{N}^*$ and let $a_0,\ldots,a_n$ be real numbers with $a_0<a_1<\cdots<a_n$.

(a) Let U be the $\mathbb{R}$-vector space of continuous *piecewise linear*[1] real valued functions os the closed interval $[a_0\,,a_n]$ in $\mathbb{R}$ with partition points $a_1,\ldots a_{n-1}$. Show that the functions $|t-a_0|\,,\ldots,\,|t-a_n|$ is a $\mathbb{R}$-basis of U. In particular, $\mathrm{Dim}_K U = n+1$.

(b) Let V be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $\mathbb{R}\to\mathbb{R}$ with partition points $a_0,\ldots,a_n$. Show that the functions $(a_0-t)_+\,, |t-a_0|\,,\ldots,\,|t-a_n|\,, (t-a_n)_+$ is a basis of V, where $f_+ := \mathrm{Max}\,(f,0)$ denote the positive part of a real valued function f. In particular, $\mathrm{Dim}_K V = n+3$.

(c) Let W be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $[a_0\,,a_n]\to\mathbb{R}$ with partitions points $a_1,\ldots,a_{n-1}$, and which vanish at both the end points a_0 and a_n. Show that there exist functions $f_1,\ldots,f_{n-1}\in W$ and the functions $g_1,\ldots,g_{n-1}\in W$ which form bases of W such that the graphs of f_i and g_i are:



(d) Let $k,m\in\mathbb{N}$ with $k<m$. The set of k-times continuously differentiable $\mathbb{R}$-valued functions on the closed interval $[a_0\,,a_n]$, which are polynomial functions of degree $\le m$ on every subinterval $[a_i\,,a_{i+1}]$, is a $\mathbb{R}$-vector space of dimension $(m-k)n+k+1$ with basis

$$1,\ (t-a_0)\,,\ldots,(t-a_0)^m,\ \big((t-a_1)_+\big)^{k+1},\ldots,\big((t-a_1)_+\big)^m,\ldots,\big((t-a_{n-1})_+\big)^{k+1},\ldots,\big((t-a_{n-1})_+\big)^m.$$

(**Remark :** The elements of this vector space are called s p l i n e f u n c t i o n s of type (m,k) on $[a_0\,,a_n]$ with partition points $a_1,\ldots,a_{n-1}$.)

S4.16 Let K be a field and $F = a_0 + a_1X + \cdots + a_nX^n \in K[X]$ be a polynomial of degree $\deg F = n$, $n\in\mathbb{N}$. Suppose that the multiples $m\cdot 1_K$, $m\in\mathbb{N}^*$, are all $\neq 0$ [2] (for example, $K=\mathbb{Q}$, $\mathbb{R}$ and $K=\mathbb{C}$ have this property). For pairwise distinct elements $\lambda_0,\ldots,\lambda_n\in K$, the polynomials $F(X-\lambda_0),\ldots,F(X-\lambda_n)\in K[X]_{n+1}$ form a K-basis of the K-vector space $K[X]_{n+1}$ of polynomials of degree $\le n$ over K. In particular, the polynomials $(X-\lambda_0)^n,\ldots,(X-\lambda_n)^n$ form a basis of $K[X]_{n+1}$. (**Hint :** Since $1,X,\ldots,X^n$ is a K-basis of $K[X]_{n+1}$, $\mathrm{Dim}_K K[X]_{n+1} = n+1$ and hence it is enough to prove the linear independence of $F(X-\lambda_0),\ldots,F(X-\lambda_n)$ over K. which is proved in Exercise 3.5 (b).)

†**S4.17** Let $n\in\mathbb{N}^*$. Show that there exist a representation in $\mathbb{Q}[t]$ of the form

$$t = \sum_{k=0}^{n} \frac{a_k}{b}(t+k)^n\,, \qquad a_k\in\mathbb{Z},\ b\in\mathbb{N}^*\,.$$

Use this to deduce that there exists a natural number $\gamma(n)$ such that every natural number is a sum of $\gamma(n)$ integers of the form $\pm m^n$, $m\in\mathbb{N}$. (**Hint :** For a representation use the above Supplement S4.16. For multiples of b the last assertion directly follows from the above formula, otherwise apply division with remainder. — **Remarks:** Further, one can choose $\gamma(n)\le |a_0|+\cdots+|a_n|+[b/2]$. In particular, one can even have $\gamma(2)=3$ and $\gamma(3)=5$, where it is still unknown whether or not $\gamma(3)=4$. Since 6 and 14 can not be written in the form $m_1^2\pm m_2^2$, the equality $\gamma(2)=2$ is not enough. — The *Two-Square Theorem* (Fermat-Euler) describes exactly those natural numbers $m\in\mathbb{N}$ which can not be expressed in the form $m_1^2\pm m_2^2$. Since 4 and 5 can not be expressed in the form $m_1^3\pm m_2^3\pm m_3^3$, as one sees this by computing modulo 9, it follows that the equality $\gamma(3)=3$ is not enough. – Moreover, it is conjectured by E. Waring[3] (and D. Hilbert proved it, even sharper) that: *There exists a natural number $g(n)$ such that every natural number is sum of $g(n)$*

[1] Let $n\in\mathbb{N}^*$ and let $a_0,\ldots,a_n$ be real numbers with $a_0<a_1<\cdots<a_n$. A continuous real valued function $f:[a_0,a_n]\to\mathbb{R}$ is called p i e c e w i s e l i n e a r w i t h p a r t i t i o n p o i n t s $a_0,\ldots,a_n$ if $f\big|[a_i,a_{i+1}]\to\mathbb{R}$ is linear for every $i=1,\ldots,n-1$. — A real valued function $f:[a,b]\to\mathbb{R}$ defined on the closed interval $[a,b]\subseteq\mathbb{R}$ is called l i n e a r if there exist $\lambda,\mu\in\mathbb{R}$ such that $f(t)=\lambda t+\mu$ for every $t\in[a,b]$.

[2] In this one also says that K has the c h a r a c t e r i s t i c 0.

[3] An English mathematician E. Waring stated without proof that every number is the sum of 4 squares, of 9 cubes, of 19 biquadrates, and so on in *Meditationes algebraicae* (1770), 204-205 and Lagrange proved that $g(2)=4$ (*Lagrange's four-square theorem*) later in the same year. It is very improbable that Waring had any sufficient grounds for his assertion and it was until more than 100 years later that Hilbert first proved (even sharper assertion) that it is true. Hilbert's

natural numbers of the form m^n, $m \in \mathbb{N}$. In other words: *To determine , for a given positive natural number* n, *there is a natural number* $g(n)$ *such that the equation* $a = x_1^n + \cdots x_{g(n)}^n$ *has a solution in* $\mathbb{N}^{g(n)}$ *for every* $a \in \mathbb{N}$. This is known as the Waring's Problem. Previous writers had proved its existence when $n = 3,4,5,6,7,8$ and 10, but its value $g(n)$ is determined only for $n = 3$. The value $g(n)$ is now known for all n. For example, $g(2) = 4$, $g(3) = 9$, $g(4) = 19$, $g(5) = 37$. Except for $g(2)$ and $g(3)$, the known proofs of these results involve much more complicated methods and use heavily the theory of functions of complex variable.)

S4.18 Let K be a field and let $a_0,\ldots,a_m \in K$, $a_m \neq 0$. Show that the subset

$$V(a_0,\ldots,a_m) := \{(x_n \in K^{\mathbb{N}} \mid a_0x_m + a_1x_{m+1} + \cdots + a_{m-1}x_{n+m-1} + a_mx_{n+m} = 0 \text{ for all } n \in \mathbb{N}\}$$

is a subspace of $K^{\mathbb{N}}$ of the dimension n. (**Remark:** We say that a sequence $(x_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ satisfy the (linear) recursion equation with (recursion) polynomial $\alpha(X) = a_0 + a_1X + \cdots + a_mX^m \in K[X]$ if $(x_n)_{n\in\mathbb{N}} \in V(a_0,\ldots,a_m)$. If K is algebraically closed (for example, if $K = \mathbb{C}$), then one can also find a K-basis of $V(a_0,\ldots,a_m)$ in by using the zeros of the polynomial $\alpha(A)$.)

†**S4.19** **(a)** Let $U \subseteq K^n$ be a subspace of dimension m. Then there exists uniquely determined basis of U of the form

$$\begin{aligned} v_1 &= (*,\ldots,*,1,0,\ldots,0) \in K^n\,,\\ v_2 &= (*,\ldots,*,0,*,\ldots,*,1,0,\ldots,0) \in K^n\,,\\ &\cdots\cdots\cdots\cdots\\ v_m &= (*,\ldots,*,0,*,\ldots,*,0,*,\ldots,*,0,\ldots,1,0,\ldots,0) \in K^n\,, \end{aligned}$$

where in the vectors v_j, $j = 1,\ldots,m$, at the positions $*$ there are elements in K which are uniquely determined by U and 1 is at the positions d_j with $1 \le d_1 < d_2 < \cdots < d_m \le n$, these positions are also uniquely determined by U. (**Remarks:** The set

$$\mathrm{G}_K(m,n) := \{U \subseteq V \mid U \text{ is a } K\text{-subspace of } V \text{ with } \mathrm{Dim}_K U = m\}$$

of all m-dimensional subspaces of K^n is called the Grassmann-Mannifold of the type (m,n) over K. The aim of this is Exercise is to give a partition of $\mathrm{G}_K(m,n)$ into subsets $\sigma(d_1,\ldots,d_m)$, where $(d_1,\ldots,d_m)$ runs through the subset

$$\{\{d_1,\ldots,d_m\} \in \mathfrak{P}(\{1,\ldots,n\}) \mid 1 \le d_1 < \cdots < d_m \le n\}$$

of $\mathfrak{P}(\{1,2,\ldots,n\})$ of cardinality $\binom{n}{m}$. The subspace corresponding to $\sigma := \sigma(d_1,\ldots,d_m)$ is then parameterized by the tuple in K^{k_σ} where

$$k_\sigma := (d_1 - 1) + \cdots + (d_m - m) = \sum_{j=1}^{m} d_j - \binom{m+1}{2}.$$

$\sigma(d_1,\ldots,d_m)$ is called a Schubert-cell of the dimension

$$k_\sigma = \sum_{j=1}^{m} d_j - \binom{m+1}{2}$$

in $\mathrm{G}_K(m,n)$. Further, $\sigma(1,\ldots,m)$ respectively, $\sigma(n-m+1,\ldots,n)$ are the only Schubert-cells of the minimal dimension 0 respectively, the maximal dimension $m\ell$, $\ell := n-m$. — The definition of the Schubert-cells and their notation is not uniform in the literature. If we put $\delta_j := d_j - j$, $j = 1,\ldots,m$, then a sequence $0 \le \delta_1 \le \cdots \le \delta_m \le \ell$ and the corresponding cell has the dimension $\delta_1 + \cdots + \delta_m$. Therefore : *For a given* $k \in \mathbb{N}$, *the number of Schubert-cells of dimension* k *is the number* $p(k;m,\ell)$ *of partitions of the number* k *with at most* m *positive natural numbers* $\le \ell$. For example, if K is a finite field with q elements, then

$$|\mathrm{G}_K(m,n)| = \sum_{k=0}^{m\ell} p(k;m,\ell)\, q^k\,.$$

Moreover, *this sum is equal to the value* $G_m^{[n]}(q)$ *of the Gauss-polynomial* $G_m^{[n]}$ *at the place* q. One can use this result and the Identity-Theorem for polynomials to give a combinatorial proof of the following equality of polynomials:

$$G_m^{[n]}(T) = \sum_{k=0}^{m\ell} p(k;m,\ell)\,T^k = \frac{(T^n-1)\cdots(T^{n-m+1}-1)}{(T^m-1)\cdots(T-1)}\,, \quad \ell = n-m\,.)$$

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proof of the existence of $g(n)$ for every n was published in *Göttinger Nachrichten* (1909), 17-36 and *Math.Annalen*, **67** (1909), 281-305.

(b) Compute the bases described in part (a) for the subspaces U and for W given in the Supplement S4.3.

S4.20 Let V be an n-dimensional vector space over a field K and let U and W be K-subspaces of V of dimensions p and q, respectively. Which numbers can occur as the dimensions of $U \cap W$?

S4.21 Let $V = Kx_1 + \cdots + Kx_n + Kx_{n+1}$ be a K-vector space, W be a K-subspace of V with $W \not\subseteq V' := Kx_1 + \cdots + Kx_n$ and let y be an arbitrary vector in $W \setminus V'$. Then show that

$$W = W \cap V' + Ky\,.$$

By induction on n it follows directly that every subspace of a K-vector space which a generating system consisting of n vectors, itself has a generating system consisting of at most n vectors.

S4.22 Let $v_1, \ldots, v_n$ be a basis of the n-dimensional K-vector space V, $n \geq 1$, and H be a hyperplane in V. Show that there exist i_0, $1 \leq i_0 \leq n$, and elements $a_i \in K$, $i \neq i_0$ such that $v_i - a_i v_{i_0}$, $i \neq i_0$ is a basis of H. In which case for *every* $i_0 \in \{1, \ldots, n\}$ there are such elements $a_i \in K$?

S4.23 Let V be a finite dimensional vector space over a field K and V_i, $i \in I$, be a family of K-subspaces of V. Then there exists a finite subset $J \subseteq I$ such that

$$\bigcap\nolimits_{i \in I} V_i = \bigcap\nolimits_{j \in J} V_j \quad \text{and} \quad \sum\nolimits_{i \in I} V_i = \sum\nolimits_{j \in J} V_j\,.$$

S4.24 Let K be a field,, V be a n-dimensional K-vector space and

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V$$

be a sequence of K-subspaces with $\mathrm{Dim}_K V_i \leq i$ for $i = 0, \ldots, n$. Then show that there is a flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = V$$

in V with $V_i \subseteq W_i$ for all $i = 1, \ldots, n$. (A maximal strictly ascending chain

$$0 = W_0 \subsetneq W_1 \subset \cdots \subsetneq W_n = V$$

of K-subspaces (for which necessarily $\mathrm{Dim}_K W_i = 1$, $i = 0, \ldots, n$) is called a f l a g o f V. For such a flag of V, if $w_i \in W_i \smallsetminus W_{i-1}$, $i = 1, \ldots, n$, then $W_i = \sum_{j=1}^{i} K w_j$ and $w_1, \ldots, w_n$ is a K-basis of V.)

S4.25 Let V be a vector space over a field K which is not finite dimensional over K. Construct an infinite strictly ascending $U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n \subsetneq U_{n+1} \subsetneq \cdots$ and an infinite strictly descending $W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_n \supsetneq W_{n+1} \subsetneq \cdots$ of K-subspaces of V.

S4.26 Let V be a finite dimensional K-vector space. If V_i, $i \in I$, are subspaces of V with

$$\mathrm{Codim}_K \bigcap\nolimits_{i \in I} V_i = m \in \mathbb{N}\,,$$

then show that there exists a finite subset $J \subseteq I$ with $|J| \leq m$ and $\bigcap_{i \in I} V_i = \bigcap_{i \in J} V_i$. (**Remark :** See also Exercise 4.2. — This statement also hold even if V is not finite dimensional, if we put $\mathrm{Codim}_K U := \mathrm{Dim}_K V/U$, where V/U denote the *quotient space* of V by U.)

S4.27 Let $L\,|\,K$ be an extension of fields. Further, let V_L is an L-vector space with L-basis $x_1, \ldots, x_n$ and $V := Kx_1 + \ldots + Kx_n \subseteq V_L$. (For example : $V_L := L^n$; $x_1, \ldots, x_n$ is the standard basis ; $V = K^n$.)

(a) Show that $y_1, \ldots, y_m \in V$ are K-linearly independent (resp. form a K-generating system of V, resp. form a K-basis of V) if and only if they are L-linearly independent (resp. form a L-generating system of V_L, resp. form a L-basis of V_L).

(b) Let U be a K-subspace of V and let U_L denote the L-subspace of V_L generated by U. Then show that $\mathrm{Dim}_K U = \mathrm{Dim}_L U_L$ and $U = V \cap U_L$. Further, if $W \subseteq V$ is an another K-subspace of V, then $U \subseteq W$ (resp. $U = W$) if and only if $U_L \subseteq W_L$ (resp. $U_L = W_L$).

(c) Prove the analogous assertions in the case when V_L is not finite dimensional.

S4.28 Let K be a field and let M be a maximal K-linear independent subset consisting of the 0-1-sequences in $K^{\mathbb{N}}$. Show that the cardinality of M is the cardinality of the continuums. (One may assume that K is a prime field, i. e., either $K = \mathbb{Z}/\mathbb{Z}\,p$ for some prime number p, or $K = \mathbb{Q}$. Use countability of K and cardinality argument to show that the dimension of the K-subspace generated by 0-1-sequences $K^{\mathbb{N}}$ is the cardinality of the continuums.)