MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

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Tel : +91-(0)80-2293 3212/09449076304 **E-mails :** patil@math.iisc.ernet.in

Lectures : Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 (**if LH-1 is not free**) / LH-1

Seminars : Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

Final Examination : Tuesday, December 05, 2017, 09:00–12:00

Evaluation Weightage : Assignments : 20% **Seminars :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

6. Integral Extensions

6.1 Let $A\subseteq B$ be an extension of rings and let $x\in B^\times$. Show that

(a) $x\in B$ is integral over A if and only if $x^{-1}\in A[x]$.

(b) $A[x]\cap A[x^{-1}]$ is integral over A. (**Hint:** If $y=a_0+a_1x+\cdots+a_nx^n=b_0+b_1x^{-1}+\cdots+b_mx^{-m}$ where $m,n\in\mathbb{N}^+$, $a_0,\ldots,a_n,b_0,\ldots,b_m\in A$. The A- submodule of B generated by $1,x,\ldots,x^{m+n+1}$ is a faithful A-module.)

(c) If B is integral over A, then $B^\times\cap A=A^\times$ and $x^{-1}\in A[x]$ for all $x\in B^\times$.

6.2 For a monic polynomial $F=X^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0\in A[X]$, let

$$\mathfrak{A}_F := \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{pmatrix}$$

be the *companion matrix* of F. Further, for matrices $\mathfrak{A}=(a_{ij})\in \mathrm{M}_{I,J}(A)$, $\mathfrak{B}=(b_{rs})\in\mathrm{M}_{R,S}(A)$, let $\mathfrak{A}\otimes\mathfrak{B}$ be their *Kronecker product*[1] $(a_{ij}b_{rs})\in\mathrm{M}_{I\times R,J\times S}(A)$.

Show that : if x,y are integral elements of an A-algebra B with integral equations $F(x)=0$ and $G(y)=0$, $F,G\in A[X]$ are monic polynomials of degrees m and n respectively. Then $\chi_{\mathfrak{A}_F\otimes\mathfrak{A}_G}(xy)=0$ is an integral

[1] **Kronecker product and Tensor product** We use the following exercises from linear algebra over an arbitrary commutative ring A.

(1) For matrices $\mathfrak{A}=(a_{ij})\in\mathrm{M}_{I,J}(A)$, $\mathfrak{B}=(b_{rs})\in\mathrm{M}_{R,S}(A)$, let $\mathfrak{A}\otimes\mathfrak{B}$ be their *Kronecker product* $(a_{ij}b_{rs})\in\mathrm{M}_{I\times R,J\times S}(A)$. We can write $\mathfrak{A}\otimes\mathfrak{B}$ as block matrix in two ways :

$$\mathfrak{A}\otimes\mathfrak{B}=\big(a_{ij}\mathfrak{B}\big)_{(i,j)\in I\times J}=\big(b_{rs}\mathfrak{A}\big)_{(r,s)\in R\times S}.$$

(2) Let $f:V\to W$, $f':V'\to W'$ be A-linear maps of free A-modules and let $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}(f)$ $\mathfrak{M}^{\mathfrak{v}'}_{\mathfrak{w}'}(f')$ be the matrices of f, f' with respect to bases $\mathfrak{v}:=\{v_j\mid j\in J\}$, $\mathfrak{w}:=\{w_i\mid i\in I\}$, $\mathfrak{v}':=\{v'_{j'}\mid j'\in J'\}$, $\mathfrak{w}':=\{w'_{i'}\mid i'\in I'\}$, respectively. Then the matrix $\mathfrak{M}^{\mathfrak{v}\otimes\mathfrak{v}'}_{\mathfrak{w}\otimes\mathfrak{w}'}(f\otimes f')$ of the *tensor product* map $f\otimes f':V\otimes V'\to W\otimes W'$ with respect to bases $\mathfrak{v}\otimes\mathfrak{v}':=\{v_j\otimes v'_{j'}\mid (j,j')\in J\times J'\}$ and $\mathfrak{w}\otimes\mathfrak{w}':=\{w_i\otimes w'_{i'}\mid (i,i')\in I\times I'\}$ is the *Kronecker product* $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}(f)\otimes\mathfrak{M}^{\mathfrak{v}'}_{\mathfrak{w}'}(f')$ of the matrices $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}(f)$ and $\mathfrak{M}^{\mathfrak{v}'}_{\mathfrak{w}'}(f')$.

(a) If both f and f' are of finite rank, then $f\otimes f'$ is of finite rank and in this case $\mathrm{Rank}(f\otimes f')=\mathrm{Rank}f\cdot\mathrm{Rank}f'$. In particular, for $\mathfrak{A}\in\mathrm{M}_m(K)$ and $\mathfrak{A}'\in\mathrm{M}_n(K)$, we have : $\mathrm{Rank}(\mathfrak{A}\otimes\mathfrak{A}')=\mathrm{Rank}(\mathfrak{A})\cdot\mathrm{Rank}(\mathfrak{A}')$.

(b) Let V and V' be free A-modules of finite ranks $m:=\mathrm{Rank}_A V$ and $n:=\mathrm{Rank}_A V'$, respectively, $f\in\mathrm{End}_A V$, $f'\in\mathrm{End}_A V'$ and let $\chi_f=\prod_{i=1}^m(X-\lambda_i)$, $\chi_{f'}=\prod_{j=1}^n(X-\mu_j)$. Then

$$\chi_{f\otimes f'}=\prod_{i,j}(X-\lambda_i\mu_j),\quad \mathrm{Tr}(f\otimes f')=\mathrm{Tr}(f)\cdot\mathrm{Tr}(f')\quad\text{and}\quad \mathrm{Det}(f\otimes f')=(\mathrm{Det}\,f)^n\cdot(\mathrm{Det}\,f')^m.$$

(**Hint:** We may assume that f and f' are triangulable. Let $f=d+n$ and $f'=d'+n'$ be the additive canonical decomposition into diagonal and nilpotent operators, repsectively. Then $f\otimes f'=(d\otimes d')+(d\otimes n'+n\otimes d'+n\otimes n')$ is the additive canonical decomposition of $f\otimes f'$ into diagonal and nilpotent operators. To prove the formulas for trace and determinant, use $f\otimes f'=(f\otimes \mathrm{id}_{V'})\circ(\mathrm{id}_V\otimes f')$ and the Exercise 1. above.)

(c) In particular, the eigenvalues of $f\otimes f'$ are the product of the eigenvalues of f with the eigenvalues of f' (with equation for the product xy and $\chi_{\mathfrak{A}_F\otimes\mathfrak{E}_n+\mathfrak{E}_n\otimes\mathfrak{A}_G}(x+y)=0$ integral equation for the sum $x+y$ and both have degree mn, where $\mathfrak{E}_n$ denote the $n\times n$ identity matrix.

6.3 **(a)** In the matrix ring $\mathrm{M}_2(\mathbb{Q})$ give two elements which are integral over $\mathbb{Z}$, but neither their sum nor their product are integral over $\mathbb{Z}$. (**Hint :** Consider the unipotent matrices $\mathfrak{E}_2+\mathfrak{N}$, where $\mathfrak{E}_2$ is the identity matrix and $\mathfrak{N}$ is a nilpotent matrix.)

(b) Let K be a field and let $A:=K[Y^kX^{k+1}\mid k\in\mathbb{N}]$ be the K-subalgebra of the polynomial algebra $K[X,Y]$ generated by monomials $Y^kX^{k+1}, k\in\mathbb{N}$. Show that $A[XY]$ is contained in the finitely generated A-module, but XY is not integral over A.

6.4 Let K be a field of characteristic $\neq 2$ and let ${}^2K^\times:=\{x^2\mid x\in K^\times\}$ be the group of non-zero squares Then the residue group $K^\times/{}^2K^\times$ is called the *quadratic residue class group* of K. (Every element of $K^\times/{}^2K^\times$ has self inverse and hence $K^\times/{}^2K^\times$ is a vector space over $\mathbb{F}_2$.)

(a) Show that : for $D\in K^\times\setminus{}^2K^\times$, $K[\sqrt{D}]:=K[X]/(X^2-D)$, $\sqrt{D}:=x=$ the residue class of X, is a quadratic field extension of K and the map $K[\sqrt{D}]\mapsto D\cdot{}^2K^\times$ induces a bijective map on the set of K-algebra isomorphism classes of the quadratic field extensions of K onto the set of non-zero elements of $K^\times/{}^2K^\times$.

(b) Let K be the quotient field of the factorial ring A and let $p_i, i\in I$, be a representative system for the associative classes of the prime elements of A. Show that :

$$K^\times/{}^2K^\times\cong(A^\times/{}^2A^\times)\times\mathbb{F}_2^{(I)}.$$

(c) For the following K give a (cannonical) representative system for the isomorphism classes of the quadratic field extensions of K : (1) K is a finite field of charateristic $\neq 2$. (2) $K=\mathbb{R}$ or $K=\mathbb{C}$. (3) $K=\mathbb{Q}$. (4) $K=k(X)=\mathrm{Q}(k[X])=$ the rational function field in one variable over a field k of characteristic $\neq 2$. (5) $K=\mathbb{Q}_p$ the field of p-adic numbers. (6) $K=k((X))=\mathrm{Q}(k[[X]])=$ the field of formal Laurent series over a field k of characteristic $\neq 2$.

6.5 Let A and p_i $i\in I$ be as in the Exercise 6.4 (b). Let $J\subseteq I$ be a finite subset and let $p_J:=\prod_{i\in I}p_i$, further, let $\varepsilon\in A^\times$, $D:=\varepsilon p_J$. Assume that either $J\neq\emptyset$ or $\varepsilon\notin{}^2A^\times$ i.e. $D\notin{}^2A$. Let L be the quadratic extension $K[\sqrt{D}]$ of $K:=\mathrm{Q}(A)=$ the quotient field of A and let B be the integral closure of A in L. Show that :

(a) The elements of B are precisely

$$\frac{a+b\sqrt{D}}{2},\quad a,b\in A,\, a^2-b^2D\in 4A.$$

In particular, $A[\sqrt{D}]=A+A\sqrt{D}\subseteq B\subseteq\frac{1}{2}A[\sqrt{D}]$ and $B=A[\sqrt{D}]$, if $2\in A^\times$.

(b) If $D\in 2A$, then $B=A[\sqrt{D}]$, i.e. $1,\sqrt{D}$ is an A-basis of B.

(c) If $D\equiv 1 \bmod 4A$, then $1,\ \omega:=(1+\sqrt{D})/2$ is a A-basis of B.

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multiplicities). Further, if $f\neq 0$ and $f'\neq 0$, then $f\otimes f'$ is diagonalisable if and only if both the components f and f' are diagonalisable. Further, for $\mathfrak{A}\in\mathrm{M}_m(K)$ and $\mathfrak{A}'\in\mathrm{M}_n(K)$ we have :

$$\mathrm{Tr}(\mathfrak{A}\otimes\mathfrak{A}')=\mathrm{Tr}(\mathfrak{A})\cdot\mathrm{Tr}(\mathfrak{A}')\quad\text{and}\quad\mathrm{Det}(\mathfrak{A}\otimes\mathfrak{A}')=(\mathrm{Det}\,\mathfrak{A})^n\cdot(\mathrm{Det}\,\mathfrak{A}')^m.$$