Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005

Lectures: Tuesday/Thursday 18:15–19:15 ; LH-1, Department of Mathematics

1. The Set of Natural Numbers — Peano's axioms

Giuseppe Peano†
(1858-1932)

1.1. a). (First principle of induction) Using the third axiom of Peano prove the following: *Suppose that for each natural number $n \in \mathbb{N}$, we have associated a statement $\mathrm{S}(n)$. Assume that the following conditions are satisfied*:

(i) $\mathrm{S}(0)$ *is true.* (Begining of Induction)

(ii) *For every $n \in \mathbb{N}$, $\mathrm{S}(n+1)$ is true whenever $\mathrm{S}(n)$ is true.* (Inductive step)

Then $\mathrm{S}(n)$ is true for all $n \in \mathbb{N}$. (**Hint:** Let $M := \{n \in \mathbb{N} \mid \mathrm{S}(n) \text{ is true}\} \subseteq \mathbb{N}$. Then $0 \in M$ by the hypothesis (i). Furher, by hypothesis (ii) if $n \in M$, then $n+1 \in M$. Therefore $M = \mathbb{N}$ by the axiom third of Peano.) (**Remark:** The following variant is also used very often: *Let $n_0 \in \mathbb{N}$. Suppose that for every natural number $n \geq n_0$, we have associated a statement $\mathrm{S}(n)$. Assume that $\mathrm{S}(n_0)$ is true and for every $n \geq n_0$ $\mathrm{S}(n+1)$ is true whenever $\mathrm{S}(n)$ is true. Then $\mathrm{S}(n)$ is true for all $n \geq n_0$.* For the proof consider the set $M := \{n \in \mathbb{N} \mid n < n_0\} \cup \{n \in \mathbb{N} \mid n \geq n_0 \text{ and } \mathrm{S}(n) \text{ is true }\}$.)

b). Using the first principle of induction prove the following basic property of $\mathbb{N}$:

(Minimum Principle) *Every non-empty subset M of $\mathbb{N}$ has a smallest element*, i.e., *there exists an element $m_0 \in M$ such that $m_0 \leq m$ for all $m \in M$.* (**Hint:** For $n \in \mathbb{N}$, let $\mathrm{S}(n)$ be the following statement: *If M contains a natural number m with $m \leq n$, then M has a smallest element.* By using induction show that the statement $\mathrm{S}(n)$ is true for all n.) (**Remark:** The minimum principle for $\mathbb{N}$ is also known as the well-ordering property of $\mathbb{N}$.)

c). The above well-ordering property of $\mathbb{N}$ is the basis of the following *second principle of induction*:

(Second principle of induction) *Suppose that for each natural number $n \in \mathbb{N}$, we have associated a statement $\mathrm{S}(n)$. Assume that for every $n \in \mathbb{N}$, if the $\mathrm{S}(m)$ is true for all $m < n$, then $\mathrm{S}(n)$ is also true. Then $\mathrm{S}(n)$ is true for all $n \in \mathbb{N}$.* (**Hint:** Let $M := \{n \in \mathbb{N} \mid \mathrm{S}(n) \text{ is NOT true}\} \subseteq \mathbb{N}$. Then show that $M = \emptyset$.)

1.2. Latin squares could be used by dating services to organize meetings between a number n of girls and the same number n of boys. Having met all the boys, each girl comes up with a list of boys she would not mind marrying. The dating service is faced now with the task of arranging marriages so as to satisfy each girl preferences. Call the set of boys listed by the i-th girl Y_i. The problem is then to pick boys, one from each list, without selecting the same boy more than once. An abstract formulation of this problem the following theorem:

(Marriage Theorem) *Let Y_i, $i \in I$ be a finite family of sets, i.e. I is a finite set. Suppose that for every subset J of I, the set $Y_J := \cup_{j \in J} Y_j$ contains at least $\mathrm{card}(J)$ elements. Then there exists an injective choice function $f : I \to Y_I$ with $f(i) \in Y_i$ for every $i \in I$.* (**Hint:** Use induction on the cardinality $\mathrm{card}(I)$ of I.)

(**Remarks:** In mathematics, the *marriage theorem* (1935), usually credited to mathematician PHILIP HALL, is a combinatorial result that gives the condition allowing the selection of a distinct element from each of a collection of subsets.

The standard example (somewhat dated at this point) of an application of the marriage theorem is to imagine two groups of n men and women. Each woman would happily marry some subset of the men; and any man would be happy to marry a woman who wants to marry him. If we let M_i be the set of men that the i-th woman would be happy to marry, then each woman can happily marry a man if and only if the collection of sets $\{M_i\}$ meets the *marriage condition* (=hypothesis in the Marriage theorem).

The theorem has many other interesting "non-marital" applications. For example, take a standard deck of cards, and deal them out into 13 piles of 4 cards each. Then, using the marriage theorem, we can show that it is possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (ace, 2, 3, ..., queen, king).

More abstractly, let G be a group, and H be a finite subgroup of G. Then the marriage theorem can be used to show that there is a set X such that X is an SDR[1]) for both the set of left cosets and right cosets of H in G.

This can also be applied to the problem of Assignment: Given a set of n employees, fill out a list of the jobs each of them would be able to preform. Then, we can give each person a job suited to their abilities if, and only if, for every value of $k = 1, \ldots n$, the union of any k of the lists contains at least k jobs.)

1.3. Proofs by induction are very common in Mathematics and are undoubtedly familer to the reader. One also encounters quite frequently – without being conscious of it – definitions by induction or recursion. For example, powers of a non-zero real number a^n are defined by $a^0 = 1, a^{r+1} = a^r a$. Definition by induction is not as trivial as it may appear at first glance. This can be made precise by the following well-known recursion theorem proved by DEDEKIND :

a). (Recursion Theorem) *Let* X *be a non-empty set and let* $F : X \to X$ *be a map. For* $a \in X$*, there exists a unique (sequence in* X*) map* $f : \mathbb{N} \longrightarrow X$ *such that* (i) $f(0) = a$ *and* (ii) $f(s(n)) = F(f(n))$ *for all* $n \in \mathbb{N}$*, i.e., the following diagramm is commutative.*

$$\begin{tikzcd} \mathbb{N} \arrow[r, "s"] \arrow[d, "f"'] & \mathbb{N} \arrow[d, "f"] \\ X \arrow[r, "F"] & X \end{tikzcd}$$

(**Hint:** Uniqueness of f is clear by induction. For existence, put $I_n := \{0, 1, \ldots, n\}$. By induction show that the following statement $\mathrm{S}(n)$ is true for all $n \in \mathbb{N}$. $\mathrm{S}(n)$: *There exists a unique map* $f_n : I_n \to X$ *such that* $f_n(0) = a$ *and* $f(r+1) = F(f(r))$ *for every* $r \in \mathbb{N}$ *with* $r < n$. For arbitrary natural numbers $m, n \in \mathbb{N}$ with $m \leq n$, we then have $f_m = f_n | I_m$. Therefore $f_n(n) = F(f_n(n-1)) = F(f_{n-1}(n-1))$ for all $n \geq 1$. Now, define f by $n \mapsto f_n(n)$.) (**Remark:** One might be tempted to say that one can *define* inductively by conditions (i) and (ii). However, this does not make sense since in talking about a function on $\mathbb{N}$ we must have an à priori definition of $f(n)$ for every $n \in \mathbb{N}$. A proof of the existence of f must use *all* of PEANO's axioms. See the example illustrating this in b) below.)

b). (HENKIN) Let $N = \{0, 1\}$ and define the map $s_N : N \to N$ by $s_N(0) := 1$ and $s_N(1) := 1$. Show that (N, s_N) satifies PEANO's axioms 1 and 3 but not 2. Show that the recusion theorem breaks down for (N, s_N). (**Hint:** Let $F : N \to N$ be the map defined by $F(0) = 1$ and $F(1) = 0$. Show that there is no map $f : N \to N$ satisfying $f(0) = 0$ and $f(s_N(a)) = F(f(a))$ for all $a \in N$.)

c). (Iteration of maps) Let X be a set, $\Phi : X \to X$ be a map, i.e., $\Phi \in X^X$. and let $F : X^X \to X^X$ be the map defined by $\Psi \mapsto \Phi \circ \Psi$. Then there exists a sequence $f : \mathbb{N} \to X^X$ in X^X such that $f(0) = \mathrm{id}_X$ and $f(n+1) = F(f(n)) = \Phi \circ f(n)$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ the map $f(n) : X \to X$ is called the n-th iterate of Φ and is denoted by Φ^n. Note that $\Phi^0 = \mathrm{id}_X, \Phi^{n+1} = \Phi^n \circ \Phi$ for all $n \in \mathbb{N}$. Further, $(\mathrm{id}_X)^n = \mathrm{id}_X$ for $n \in \mathbb{N}$.

d). Show that addition $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and multiplication $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ on $\mathbb{N}$ can be defined by using the recursion theorem. Further, verify the standard properties $+$ and $\cdot$, e.g., existence of identity element, associativity, commutativity, distributive laws, cancellation laws, monotonicity etc. (**Hint:**

[1]) Let Y_i $i \in I$ be a family of subsets of a set X. A set of distinct representatives (sometimes abbreviated as an SDR) is a subset $Z = \{z_i \mid i \in I\}$ of pairwise distinct elements of X, i.e., $\mathrm{card}(Z) = \mathrm{card}(I)$ and with the property that: $x_i \in Y_i$ for all $i \in I$.

For $+$ apply recursion theorem to $X=\mathbb{N}$ $F=s$ and $a=m\in\mathbb{N}$ to get the unique map $s_m:\mathbb{N}\to\mathbb{N}$ such that $s_m(0)=m$ and $s_m(s(n))=s(s_m(n)$ for all $n\in\mathbb{N}$. Now, define $m+n:=s_m(n)$. Note that $m+0=s_m(0)=m$ and $m+s(n)=s_m(s(n))=s(s_m(n))$. Further, note that for $m\in\mathbb{N}$, the map $s_m:\mathbb{N}\to\mathbb{N}$ is the m-th iterate (see b)) $s^m=\underbrace{s\circ s\circ\cdots\circ s}_{m\text{-times}}$ of the successor map s. For $m,n\in\mathbb{N}$, define the multiplication $m\cdot n:=s_n^m(0)=(s^n)^m(0)$.)

e). Show that there exists a binary operation of exponentiation (or n-th power of m) $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $(m,n)\mapsto m^n$. Further, state and verify the standard laws of exponents. (**Hint:** For $m\in\mathbb{N}$, let $p_m:\mathbb{N}\to\mathbb{N}$ be the multiplication by m. Define $m^n:=p_m^n(1)$.)

f). (Simultaneous recursion) Let X,Y be sets and let $H:X\times Y\to X$, $K:X\times Y\to Y$ be given maps. For $(a,b)\in X\times Y$, there exist a unique maps $f:\mathbb{N}\to X$ and $g:\mathbb{N}\to Y$ such that $f(0)=a$, $g(0)=b$ and $f(n+1)=H(f(n),g(n))$, $g(n+1)=K(f(n),g(n))$ for all $n\in\mathbb{N}$.

(**Hint:** Apply recursion theorem to the set $X\times Y$, the map $F:=H\times K:X\times Y\to X\times Y$, $(x,y)\mapsto(H(x,y),K(x,y))$ and $(a,b)\in X\times Y$, to get the map $G:\mathbb{N}\to X\times Y$ such that $G(0)=(a,b)$ and $G(n+1)=F(G(n))$ for all $n\in\mathbb{N}$. Now, take $f=p\circ G$ and $q\circ G$, where $p:X\times Y\to X$ (resp. $q:X\times Y\to Y$) is the first (resp. second) projection. Using the properties of G check that f and g have the required properties.)

g). (Primitive recursion) Let X be a set, $a\in X$ and let $H:X\times\mathbb{N}\to X$ be a given map. Show that there exists a unique map $f:\mathbb{N}\to X$ such that $f(0)=a$ and $f(n+1)=G(f(n),n)$ for all $n\in\mathbb{N}$. (**Hint:** Apply the simultaneous recursion for $Y=\mathbb{N}$, $b=0$ and the map $K:X\times\mathbb{N}\to\mathbb{N}$ defined by $(x,n)\mapsto n+1$.)

h). Construct a map $f:\mathbb{N}\to\mathbb{N}$ such that $f(0)=1$ and $f(n)=1\cdot 2\cdots(n-1)\cdot n$ (the product of the first n non-zero natural numbers) for each $n>0$. (**Hint:** Use the primitive recursion to $X=\mathbb{N}$, $a=1$ and $H:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ the map defined by $H(m,n)=(n+1)\cdot m$.) (**Remark:** For each $n\in\mathbb{N}$, the natural number $F(n)$ is called factorial n and is denoted by $n!$.)

i). (n-ary operations – generalized sums and products) Let $n\in\mathbb{N}$, X be a set and let $X^{\{1,\ldots,n\}}:=X^n:=\underbrace{X\times\cdots\times X}_{n\text{- times}}$. A map $f:X^n\to X$ is called an n-ary operation on X.

Let X be a set and let $*:X\times X\to X$ be a binary operation on X. Then there exists a unique family $f_n:X^n\to X$, $n\in\mathbb{N}^*$ of n-ary operation on X such that: $f_1=\mathrm{id}_X$, $f_2=*$ and

$$f_{n+1}((x_1,\ldots,x_n,x_{n+1}))=f_n((x_1,\ldots,x_n))*x_{n+1} \text{ for all } (x_1,\ldots,x_n,x_{n+1})\in X^{n+1} \text{ and for all } n\geq 1.$$

(**Remarks:** Applying the above part to the operation of addition $+$ on $\mathbb{N}$, we have a unique family $f_n:\mathbb{N}^n\to X$, $n\in\mathbb{N}^*$ of n-ary operation on $\mathbb{N}$. For $n\in\mathbb{N}$ and $(x_1,\ldots,x_n)\in\mathbb{N}^n$, $f_n((x_1,\ldots,x_n))$ is denoted by $\sum_{i=1}^n x_i$. Therefore $\sum_{i=1}^0 x_i=0$ and $\sum_{i=1}^{n+1}x_i=\left(\sum_{i=1}^n x_i\right)+x_{n+1}$ for all $(x_1,\ldots,x_n,x_{n+1})\in\mathbb{N}^{n+1}$ and for all $n\geq 1$.

Similarly, applying the above part to the operation of multiplication $cdot$ on $\mathbb{N}$, we have a unique family $p_n:\mathbb{N}^n\to X$, $n\in\mathbb{N}^*$ of n-ary operation on $\mathbb{N}$. For $n\in\mathbb{N}$ and $(x_1,\ldots,x_n)\in\mathbb{N}^n$, $p_n((x_1,\ldots,x_n))$ is denoted by $\prod_{i=1}^n x_i$. Therefore $\prod_{i=1}^0 x_i=1$ and $\prod_{i=1}^{n+1}x_i=\left(\prod_{i=1}^n x_i\right)+x_{n+1}$ for all $(x_1,\ldots,x_n,x_{n+1})\in\mathbb{N}^{n+1}$ and for all $n\geq 1$.

For $n\in\mathbb{N}$, $(x_1,\ldots,x_n)\in\mathbb{N}^n$ and any permutation σ of $\{1,\ldots,n\}$, prove that $\sum_{i=1}^n x_i=\sum_{i=1}^n x_{\sigma(i)}$ and $\prod_{i=1}^n x_i=\prod_{i=1}^n x_{\sigma(i)}$.

Finally, applying the above part to the operation of composition X^X, we have a unique family $\Phi_n:(X^X)^n\to X^X$, $n\in\mathbb{N}^*$ of n-ary operation on X^X. For $n\in\mathbb{N}$ and $(f_1,\ldots,f_n)\in(X^X)^n$, $\Phi_n((f_1,\ldots,f_n))$ is denoted by $f_\circ f_2\circ\cdot\circ f_n$. In particular, if $f_i=f$ for every $i\geq 1$, then for $n\geq 1$ $\Phi_n((f,f,\ldots,f))=f^n$ is the n-th iterate of f (see also 1.1-b)).)

j). Let X be a set, $a\in X$, $Y:=\bigcup_{n\in\mathbb{N}}X^n$ and let $G:Y\to X$ be a map. Then there exists a unique sequence $g:\mathbb{N}\to X$ such that , $g(0)=a$ and $g(n+1)=G(g(0),g(1),\ldots,g(n))$ for all $n\in\mathbb{N}$. (**Hint:** Define the map $F:Y\to Y$ be $(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_n,G((x_1,\ldots,x_n)))$. Then by recursion theorem there exists a unique map $f:\mathbb{N}\to Y$ such that $f(0)=a$ and $f(n+1)=F(f(n))$ for all $n\in\mathbb{N}$. Now, define $g:\mathbb{N}\to X$ by $n\mapsto f(n)(n)$.)

k). (Double recursion) Let X be a set, $a \in X$ and let $F, G : X \to X$ be two maps. Then there exists a unique map $g : \mathbb{N} \times N \to X$ such that $g((0,0)) = a$,

$$g((0, n+1)) = F(g(0,n)) \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad g((m+1, n)) = G(g(m,n)) \quad \text{for all} \quad m, n \in \mathbb{N}\,.$$

Use double recursion to obtain directly the operations of addition $+$ and $\cdot$ on $\mathbb{N}$.

1.4. Let $\widetilde{N}$ be a non-empty set, $\widetilde{0} \in \widetilde{\mathbb{N}}$ and let $\widetilde{s} : \widetilde{\mathbb{N}} \to \widetilde{\mathbb{N}}$ be a map. Suppose that for each map $F : X \to X$ and each $a \in X$, there exists a unique map $\widetilde{f} : \widetilde{\mathbb{N}} \to X$ such that (i) $\widetilde{f}(\widetilde{0}) = a$ and (ii) $\widetilde{f}(\widetilde{s}(n)) = F(\widetilde{f}(n))$ for all $n \in \mathbb{N}$, i.e., the diagramm

$$\begin{tikzcd}
\widetilde{\mathbb{N}} \arrow[r, "\widetilde{s}"] \arrow[d, "\widetilde{f}"'] & \widetilde{\mathbb{N}} \arrow[d, "\widetilde{f}"] \\
X \arrow[r, "F"] & X
\end{tikzcd}$$

is commutative. Then there exists a unique bijective map $\Phi : \mathbb{N} \to \widetilde{\mathbb{N}}$ such that $\Phi(0) = \widetilde{0}$ and $\Phi(s(n)) = \widetilde{s}(\Phi(n))$ for all $n \in \mathbb{N}$, i.e., the diagramm

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "s"] \arrow[d, "\Phi"'] & \mathbb{N} \arrow[d, "\Phi"] \\
\widetilde{\mathbb{N}} \arrow[r, "\widetilde{s}"] & \widetilde{\mathbb{N}}
\end{tikzcd}$$

is commutative. (**Remark:** This exercise says that $\mathbb{N}$ is essentially unique as a set on which maps can be defined by recursion.)

1.5. (Fibonacci Sequence) The sequence f_n, $n \in \mathbb{N}$, defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$, is called the Fibonacci Sequence and its n-th term f_n is called the n-th Fibonacci number. The first few terms of the Fibonacci Sequence are $0, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. (**Remark:** How do we know such a sequence exists? The recursion theorem cannot directly justify its existence, for the value f_{n+1} for $n \geq 1$ depend not only on f_n, but uopon f_{n-1} as well. However, we can justify the simultaneous existence of the two sequences f_n and g_n satisfying: $\begin{cases} f_0 = 0,\ f_{n+1} = f_n + g_n\,, & \text{for } n \geq 0\,, \\ g_0 = 1,\ g_{n+1} = f_n\,, & \text{for } n \geq 0\,. \end{cases}$ For this we can use the simultaneous recursion by taking $(a,b) = (0,1)$, $H : \mathbb{N} \times \mathbb{N} \to N$ is the addition on $\mathbb{N}$ and $K : \mathbb{N} \times \mathbb{N} \to N$ is the first projection.)

a). For the n-th Fibonacci number prove the following explicit (Binet's Formula):

$$f_n = \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \left(\tfrac{1-\sqrt{5}}{2}\right)^n \right)$$

b). Prove the following equalities by induction: (i) $f_{n+m} = f_{n-1} f_m + f_n f_{m+1}$ for all $m \geq 0$ and all $n \geq 1$. In particular, $f_{2n} = f_n(f_{n-1} + f_{n+1}) = f_{n+1}^2 - f_{n-1}^2$ for all $n \geq 1$.

(ii) $f_n^2 = f_{n-1} f_{n+1} + (-1)^{n+1}$ for all $n \geq 1$.

(iii) $\varphi^n = f_{n-1} + f_n \varphi$, for all $n \in \mathbb{N}^*$, where $\varphi := (1 + \sqrt{5})/2$. (**Remark:** Using this equality we can define the Fibonacci-numbers f_n for all $n \in \mathbb{Z}$. We then have $f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{Z}$.)

1.6. Let X be a non-empty set which is not finite. Then there exists an injective map $\mathbb{N} \to X$. (**Hint:** Consider the set $\mathfrak{P}_{\mathrm{f}}(X) := \{A \in \mathfrak{P}(X) \mid A \text{ is finite}\}$ of all finite subsets of X. Then for every $A \in \mathfrak{P}_{\mathrm{f}}(X)$, the complement $X \setminus A$ is a non-empty subset of X and by the *axiom of choice* there exists a *choice function* $g : \mathfrak{P}_{\mathrm{f}}(X) \to \bigcup_{A \in \mathfrak{P}_{\mathrm{f}}(X)} (X \setminus A)$, i.e., $g(A) \in X \setminus A$ for every $A \in \mathfrak{P}_{\mathrm{f}}(X)$. Now, apply recursion theorem to the map $F : \mathfrak{P}_{\mathrm{f}}(X) \to \mathfrak{P}_{\mathrm{f}}(X)$ defined by $A \mapsto A \cup \{g(A)\}$, to get a sequence $f : \mathbb{N} \to \mathfrak{P}_{\mathrm{f}}(X)$ in $\mathfrak{P}_{\mathrm{f}}(X)$ such that $f(0) = \emptyset$ abd $f(n+1) = F(f(n))$ for all $n \geq 1$. let $x_n := g(f(n))$. Then it is clear that the subset $f(n)$ is contained in the subset $\{x_0, \ldots, x_{n-1}\}$. This shows that the map $\mathbb{N} \to X$, $n \mapsto x_n$ is injective. The above proof is also written shortly but less formal as: let $x_0 \in X$ be an arbitrary element. Assume that $x_0, x_1, \ldots, x_n \in X$ are already ben defined. Now, let x_{n+1} be an arbitrary element in $X \setminus \{x_0, x_1, \ldots, x_n\}$ (which is non-empty by assumtion).)

Below one can see (simple) test-exercises.

Test-Exercises

T1.1. (some arithmetic series) For all $n \in \mathbb{N}$, prove the following formulas by induction:

a). $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. **b).** $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. **c).** $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$.

d). $\sum_{k=1}^{n}(-1)^{k-1}k = \frac{1}{4}\big(1+(-1)^{n-1}(2n+1)\big)$. **e).** $\sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n+1}\cdot\frac{n(n+1)}{2}$.

f). $\sum_{k=1}^{n}(2k-1) = n^2$. **g).** $\sum_{k=1}^{n}(2k-1)^2 = \frac{n}{3}(4n^2-1)$. **h).** $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

i). $\sum_{k=1}^{n}\frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$. **j).** $\sum_{k=1}^{n}\frac{1}{4k^2-1} = \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$.

k). $\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$. **l).** $\sum_{k=1}^{n}\frac{k-1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{2n+1}{2(n+1)(n+2)}$.

T1.2. For all $n \geq 1$ prove:

a). $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right)$. **b).** $\prod_{k=2}^{n}\left(1-\frac{2}{k(k+1)}\right) = \frac{1}{3}\left(1+\frac{2}{n}\right)$. **c).** $\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \frac{2}{3}\left(1+\frac{1}{n(n+1)}\right)$.

T1.3. (Finite geometric series) For every real (or complex) number $q \neq 1$ and every $n \in \mathbb{N}$, prove that: **a).** $\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}$ **b).** $\prod_{k=0}^{n}(1+q^{2^k}) = \frac{q^{2^{n+1}}-1}{q-1}$. **c).** $\sum_{k=1}^{n} kq^k = \frac{nq^{n+2}-(n+1)q^{n+1}+q}{(q-1)^2}$.

T1.4. For all $n \geq 1$ prove: **a).** 5 divides $2^{n+1}+3\cdot 7^n$. **b).** 3 divides n^3+2n. **c).** 6 divides n^3-n. **d).** 7 divides $5^{2n+1}+2^{2n+1}$. **e).** 30 divides n^5-n. **f).** 3 divides $2^{2n}-1$. **g).** 15 divides $3n^5+5n^3+7n$. **h).** 133 divides $11^{n+2}+12^{2n+1}$. **i).** 5 divides $3^{n+1}+2^{3n+1}$.

T1.5. For the recursively defined sequences (a_n) in a), b), c) prove the given explicit representations.

a). $a_0 = 2,\ a_n = 2 - a_{n-1}^{-1}$, $n \geq 1$. Then $a_n = (n+2)/(n+1)$ for all $n \in \mathbb{N}$.

b). $a_0 = 0,\ a_1 = 1,\ a_n = \frac{1}{2}(a_{n-1}+a_{n-2})$, $n \geq 2$. Then $a_n = \frac{2}{3}\big(1-(-1)^n\frac{1}{2^n}\big)$ for all $n \in \mathbb{N}$.

c). $a_0 = 1$, $a_n = 1 + a_{n-1}^{-1}$, $n \geq 1$. Then $a_n = f_{n+2}/f_{n+1}$ for all $n \in \mathbb{N}$, where for $k \in \mathbb{N}$, f_k is the k-th Fibonacci-number (see exercise 1.5).

d). $a_0 = 1$, $a_n = \sum_{k=0}^{n-1} a_k$, $n \geq 1$. Then $a_n = 2^{n-1}$ for all $n \geq 1$.

T1.6. Let f_n, $n \in \mathbb{N}$ be the Fibonacci sequence (see exercise 1.5). Prove the following formulas:

a). $f_n + f_{n+1} + f_{n+3} = f_{n+4}$. **b).** $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$. **c).** $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

d). $f_1 - f_2 + f_3 - \cdots + (-1)^n f_{n+1} = (-1)^n f_n + 1$. **e).** $f_n < (5/3)^n$. **f).** $2^n f_n < (\sqrt{5}+1)^n$.

g). $f_n = (a^n - b^n)/\sqrt{5}$, where a and b are the positive and negative zeros of the quadratic equation $X^2 - X - 1 = 0$.

h). $\mathfrak{A}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1}\end{pmatrix}$, where $\mathfrak{A} := \begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}$.

i). $\operatorname{card}(\mathfrak{F}_n) = f_{n+2}$, where $\mathfrak{F}_n := \{A \in \mathfrak{P}(\{1,2,\ldots,n\}) \mid A$ does not contain any consecutive integers$\}$.

† **Giuseppe Peano** was born on 27 Aug 1858 in Cuneo, Piemonte, Italy and died on 20 April 1932 in Turin, Italy. Giuseppe Peano's parents worked on a farm and Giuseppe was born in the farmhouse 'Tetto Galant' about 5 km from Cuneo. He attended the village school in Spinetta then he moved up to the school in Cuneo, making the 5km journey there and back

on foot every day. His parents bought a house in Cuneo but his father continued to work the fields at Tetto Galant with the help of a brother and sister of Giuseppe, while his mother stayed in Cuneo with Giuseppe and his older brother.

Giuseppe's mother had a brother who was a priest and lawyer in Turin and, when he realised that Giuseppe was a very talented child, he took him to Turin in 1870 for his secondary schooling and to prepare him for university studies. Giuseppe took exams at Ginnasio Cavour in 1873 and then was a pupil at Liceo Cavour from where he graduated in 1876 and, in that year, he entered the University of Turin.

Among Peano's teachers in his first year at the University of Turin was D'Ovidio who taught him analytic geometry and algebra. In his second year he was taught calculus by Angelo Genocchi and descriptive geometry by Giuseppe Bruno. Peano continued to study pure mathematics in his third year and found that he was the only student to do so. The others had continued their studies at the Engineering School which Peano himself had originally intended to do. In his third year Francesco Faà di Bruno taught him analysis and D'Ovidio taught geometry. Among his teachers in his final year were again D'Ovidio with a further geometry course and Francesco Siacci with a mechanics course. On 29 September 1880 Peano graduated as doctor of mathematics.

Peano joined the staff at the University of Turin in 1880, being appointed as assistant to D'Ovidio. He published his first mathematical paper in 1880 and a further three papers the following year. Peano was appointed assistant to Genocchi for 1881-82 and it was in 1882 that Peano made a discovery that would be typical of his style for many years, he discovered an error in a standard definition.

Genocchi was by this time quite old and in relatively poor health and Peano took over some of his teaching. Peano was about to teach the students about the area of a curved surface when he realised that the definition in Serret's book, which was the standard text for the course, was incorrect. Peano immediately told Genocchi of his discovery to be told that Genocchi already knew. Genocchi had been informed the previous year by Schwarz who seems to have been the first to find Serret's error.

In 1884 there was published a text based on Genocchi's lectures at Turin. This book Course in Infinitesimal Calculus although based on Genocchi's lectures was edited by Peano and indeed it has much in it written by Peano himself. The book itself states on the title page that it is: *... published with additions by Dr Giuseppe Peano.*

Genocchi seemed somewhat unhappy that the work came out under his name for he wrote: *... the volume contains important additions, some modifications, and various annotations, which are placed first. So that nothing will be attributed to me which is not mine, I must declare that I have had no part in the compilation of the aforementioned book and that everything is due to that outstanding young man Dr Giuseppe Peano ...*

Peano received his qualification to be a university professor in December 1884 and he continued to teach further courses, some for Genocchi whose health had not recovered sufficiently to allow him to return to the University.

In 1886 Peano proved that if $f(x, y)$ is continuous then the first order differential equation $dy/dx = f(x, y)$ has a solution. The existence of solutions with stronger hypothesis on f had been given earlier by Cauchy and then Lipschitz. Four years later Peano showed that the solutions were not unique, giving as an example the differential equation $dy/dx = 3y^{2/3}$, with $y(0) = 0$.

In addition to his teaching at the University of Turin, Peano began lecturing at the Military Academy in Turin in 1886. The following year he discovered, and published, a method for solving systems of linear differential equations using successive approximations. However Emile Picard had independently discovered this method and had credited Schwarz with discovering the method first. In 1888 Peano published the book Geometrical Calculus which begins with a chapter on mathematical logic. This was his first work on the topic that would play a major role in his research over the next few years and it was based on the work of Schröder, Boole and Charles Peirce. A more significant feature of the book is that in it Peano sets out with great clarity the ideas of Grassmann which certainly were set out in a rather obscure way by Grassmann himself. This book contains the first definition of a vector space given with a remarkably modern notation and style and, although it was not appreciated by many at the time, this is surely a quite remarkable achievement by Peano.

In 1889 Peano published his famous axioms, called Peano axioms, which defined the natural numbers in terms of sets. These were published in a pamphlet Arithmetices principia, nova methodo exposita which, according to Kennedy were: *... at once a landmark in the history of mathematical logic and of the foundations of mathematics.*

The pamphlet was written in Latin and nobody has been able to give a good reason for this, other than: *... it appears to be an act of sheer romanticism, perhaps the unique romantic act in his scientific career.*

Genocchi died in 1889 and Peano expected to be appointed to fill his chair. He wrote to Casorati, who he believed to be part of the appointing committee, for information only to discover that there was a delay due to the difficulty of finding enough members to act on the committee. Casorati had been approached but his health was not up to the task. Before the appointment could be made Peano published another stunning result.

He invented 'space-filling' curves in 1890, these are continuous surjective mappings from $[0, 1]$ onto the unit square. Hilbert, in 1891, described similar space-filling curves. It had been thought that such curves could not exist. Cantor had shown that there is a bijection between the interval $[0, 1]$ and the unit square but, shortly after, Netto had proved that such a bijection cannot be continuous. Peano's continuous space-filling curves cannot be 1-1 of course, otherwise Netto's theorem would be contradicted. Hausdorff wrote of Peano's result in Grundzüge der Mengenlehre in 1914: *This is one of the most remarkable facts of set theory.*

In December 1890 Peano's wait to be appointed to Genocchi's chair was over when, after the usual competition, Peano was offered the post. In 1891 Peano founded Rivista di matematica, a journal devoted mainly to logic and the foundations of mathematics. The first paper in the first part is a ten page article by Peano summarising his work on mathematical logic up to that time.

Peano had a great skill in seeing that theorems were incorrect by spotting exceptions. Others were not so happy to have these errors pointed out and one such was his colleague Corrado Segre. When Corrado Segre submitted an article to Rivista di matematica Peano pointed out that some of the theorems in the article had exceptions. Segre was not prepared to just correct the theorems by adding conditions that ruled out the exceptions but defended his work saying that the moment of discovery was more important than a rigorous formulation. Of course this was so against Peano's rigorous approach to mathematics that he argued strongly: *I believe it new in the history of mathematics that authors knowingly use in their research propositions for which exceptions are known, or for which they have no proof...*

It was not only Corrado Segre who suffered from Peano's outstanding ability to spot lack of rigour. Of course it was the precision of his thinking, using the exactness of his mathematical logic, that gave Peano this clarity of thought. Peano pointed out an error in a proof by Hermann Laurent in 1892 and, in the same year, reviewed a book by Veronese ending the review with the comment: *We could continue at length enumerating the absurdities that the author has piled up. But these errors, the lack of precision and rigour throughout the book take all value away from it.*

From around 1892, Peano embarked on a new and extremely ambitious project, namely the Formulario Mathematico. He explained in the March 1892 part of Rivista di matematica his thinking: *Of the greatest usefulness would be the publiction of collections of all the theorems now known that refer to given branches of the mathematical sciences ... Such a collection, which would be long and difficult in ordinary language, is made noticeably easier by using the notation of mathematical logic ...*

In many ways this grand idea marks the end of Peano's extraordinary creative work. It was a project that was greeted with enthusiasm by a few and with little interest by most. Peano began trying to convert all those around him to believe in the importance of this project and this had the effect of annoying them. However Peano and his close associates, including his assistants, Vailati, Burali-Forti, Pieri and Fano soon became deeply involved with the work.

When describing a new edition of the Formulario Mathematico in 1896 Peano writes: *Each professor will be able to adopt this Formulario as a textbook, for it ought to contain all theorems and all methods. His teaching will be reduced to showing how to read the formulas, and to indicating to the students the theorems that he wishes to explain in his course.*

When the calculus volume of the Formulario was published Peano, as he had indicated, began to use it for his teaching. This was the disaster that one would expect. Peano, who was a good teacher when he began his lecturing career, became unacceptable to both his students and his colleagues by the style of his teaching. One of his students, who was actually a great admirer of Peano, wrote: *But we students knew that this instruction was above our heads. We understood that such a subtle analysis of concepts, such a minute criticism of the definitions used by other authors, was not adapted for beginners, and especially was not useful for engineering students. We disliked having to give time and effort to the "symbols" that in later years we might never use.*

The Military Academy ended his contract to teach there in 1901 and although many of his colleagues at the university would have also liked to stop his teaching there, nothing was possible under the way that the university was set up. The professor was a law unto himself in his own subject and Peano was not prepared to listen to his colleagues when they tried to encourage him to return to his old style of teaching. The Formulario Mathematico project was completed in 1908 and one has to admire what Peano achieved but although the work contained a mine of information it was little used.

However, perhaps Peano's greatest triumph came in 1900. In that year there were two congresses held in Paris. The first was the International Congress of Philosophy which opened in Paris on 1 August. It was a triumph for Peano and Russell, who attended the Congress, wrote in his autobiography: *The Congress was the turning point of my intellectual life, because there I met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument on which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. ... It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years ...*

The day after the Philosophy Congress ended the Second International Congress of Mathematicians began. Peano remained in Paris for this Congress and listened to Hilbert's talk setting out ten of the 23 problems which appeared in his paper aimed at giving the agenda for the next century. Peano was particularly interested in the second problem which asked if the axioms of arithmetic could be proved consistent.

Even before the Formulario Mathematico project was completed Peano was putting in place the next major project of his life. In 1903 Peano expressed interest in finding a universal, or international, language and proposed an artificial language "Latino sine flexione" based on Latin but stripped of all grammar. He compiled the vocabulary by taking words from English, French, German and Latin. In fact the final edition of the Formulario Mathematico was written in Latino sine flexione which is another reason the work was so little used.

Peano's career was therefore rather strangely divided into two periods. The period up to 1900 is one where he showed great originality and a remarkable feel for topics which would be important in the development of mathematics. His achievements were outstanding and he had a modern style quite out of place in his own time. However this feel for what was important seemed to leave him and after 1900 he worked with great enthusiasm on two projects of great difficulty which were enormous undertakings but proved quite unimportant in the development of mathematics.

Of his personality Kennedy writes: *... I am fascinated by his gentle personality, his ability to attract lifelong disciples, his tolerance of human weakness, his perennial optimism. ... Peano may not only be classified as a 19th century mathematician and logician, but because of his originality and influence, must be judged one of the great scientists of that century.*

Although Peano is a founder of mathematical logic, the German mathematical philosopher Gottlob Frege is today considered the father of mathematical logic.