Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005

Supplementary Lectures : Friday 18:15–19:15 ; LH-1, Department of Mathematics

R2. Convergent sequences and Completeness

Constantin Carathéodory†
(1873-1950)

R2.1. Unless otherwise stated, in the following exercises elements are in a fixed ordered field K , for example in the field of real numbers.

1). Let (x_n) be a convergent sequence with $s \geq x_n$ (resp. $s \leq x_n$) for almost all $n \in \mathbb{N}$. Then show that $s \geq \lim x_n$ (resp. $s \leq \lim x_n$) .

2). Let (x_n) be a convergent sequence with a positive (resp. negative) limiting value. Then almost all members of this sequence are also positive (resp. negative) .

3). Let (x_n) be a sequence and let (x_{n_k}) and (x_{m_k}) be subsequences of (x_n) such that every member x_n is contained in at least one of the subsequence, (i.e., every index n is contained in one of the index sequences (n_k) and (m_k)) . Show that the sequence (x_n) is convergent if and only if each of the subsequence converges to the same limiting value (and it is naturally equal to $\lim x_n$) .

4). A sequence (x_n) is a null-sequence if and only if $(|x_n|)$ is a null-sequence.

5). Let (x_n) be a null-sequence and let (y_n) be a bounded sequence. The the sequence $(x_n y_n)$ is also a null-sequence.

6). a). A sequence with all positive (resp. all negative) members (improperly) converges to ∞ (resp. $-\infty$) if and only if the reciprocal sequence is a null-sequence. A sequence (x_n) with $x_n \neq 0$ for all $n \in \mathbb{N}$ is a null-sequence if and only if the sequence $(|1/x_n|)$ converges to ∞ .

b). Let $\lim x_n = \infty$ and $\lim y_n = a \in \overline{K} \setminus \{0\}$. Then $\lim x_n y_n = \infty$, if $a > 0$, and $\lim x_n = -\infty$, if $a < 0$.

c). If the sequence (x_n) is convergent (improperly) to ∞ , and the sequence (y_n) is bounded, then the sum sequence $(x_n + y_n)$ is also (improperly) converges to ∞ .

7). Suppose that the sequences $(x_n + y_n)$ and $(x_n - y_n)$ are convergent with the limiting values α resp. β . Then sequences (x_n), (y_n) and $(x_n y_n)$ are also convergent and $\lim x_n = (\alpha+\beta)/2$, $\lim y_n = (\alpha - \beta)/2$, $\lim x_n y_n = (\alpha^2 - \beta^2)/4$.

8). Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence (not necessarily proper). For every permutation σ of $\mathbb{N}$, the sequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$ is also convergent with the same limiting value.

9). For which convergent sequences one can choose the place $n_0 \in \mathbb{N}$ in the definition 4.E.1 independent of ε (> 0) ?

R2.2. 1). Discuss the convergence of the sequences

$$\frac{(n+1)(n^2-1)}{(2n+1)(3n^2+1)}\,;\quad \frac{n+1}{n^2+1}\,;\quad \frac{4^n+1}{5^n}\,;\quad \frac{1}{n^2} + (-1)^n \frac{n^2}{n^2+1}$$

and determine their limiting values.

2). Let $t \longmapsto f(t) := \sum_{i=0}^{k} a_i t^i$ and $t \longmapsto g(t) := \sum_{j=0}^{m} b_j t^j$ be polynomial functions with $a_i, b_j \in \mathbb{R}$, $a_k \neq 0$, $b_m \neq 0$. For all $n \geq n_0$, assume that $g(n) \neq 0$. Show that the sequence $f(n)/g(n)$, $n \geq n_0$, is defined and we have:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0\,, & \text{if } k < m\,,\\ a_m/b_m\,, & \text{if } k = m\,,\\ \infty\,, & \text{if } k > m \text{ and } a_k/b_m > 0\,,\\ -\infty\,, & \text{if } k > m \text{ and } a_k/b_m < 0\,. \end{cases}$$

3). Determine $\lim_{n\to\infty}(n-\sqrt{n})/(n+\sqrt{n}+1)$ and $\lim_{n\to\infty}(\sqrt{n}+1)/(n+1)$.

4). Determine the following limiting values:

$$\lim_{n\to\infty}\big(\sqrt{n+1}-\sqrt{n}\,\big)\,;\quad \lim_{n\to\infty}\sqrt{n}\,\big(\sqrt{n+a}-\sqrt{n}\,\big)\,,\ a\in\mathbb{R},\ n\geq |a|\,;$$

$$\lim_{n\to\infty}\Big(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\Big)n\,;\quad \lim_{n\to\infty}\Big(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\Big)\,;\quad \lim_{n\to\infty} n\,\Big(\sqrt{1+\frac{1}{n}}-1\Big),\ n\geq 1\,.$$

5). Let $a > 0$. Then show that $\lim_{n\to\infty}\sqrt[n]{a} = 1$. (**Hint:** If $a \geq 1$, write $\sqrt[n]{a} = 1 + h_n$ and apply the Bernoulli's inequality (see Exercise R1.5-1)) or use the fact that if $a \geq 1$, then the sequence is monotone decreasing and hence converges to $x \geq 1$, on the other hand $x = \lim \sqrt[n]{a} = \lim(\sqrt[2n]{a})^2 = x^2$.)

6). Show that $\lim \sqrt[n]{n} = 1$. (**Hint:** Use arguments similar to the exercise 5) above. For $n \geq 3$, the sequence is monotone decreasing.)

R2.3. 1). Let (x_n) be a sequence of non-zero real numbers.

a). If there exists a real number q with $0 < q < 1$ and $|x_{n+1}/x_n| \leq q$ for almost all n, then $\lim x_n = 0$.

b). If there exists a real number q with $q > 1$ and $|x_{n+1}/x_n| \geq q$ for almost all n, then $\lim |x_n| = \infty$.

c). Show that $\lim_{n\to\infty}\binom{n}{k}/2^n = 0$ for every $k \in \mathbb{N}$.

2). Let $a_1, \ldots, a_m \in \mathbb{R}_+, m \geq 1$. Show that $\lim_{n\to\infty}\sqrt[n]{a_1^n + \cdots + a_m^n} = \operatorname{Max}(a_1, \ldots, a_m)$.

3). Let (x_n) be (eventually improper) convergent sequence of real numbers with $\lim x_n = x$.

a). The sequence $a_n := \frac{1}{n}(x_1 + \cdots + x_n)$, $n \geq 1$, of arithmetic mean converges to x.

b). Suppose that $x_n > 0$ for all n. Then the sequence $h_n := \dfrac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$, $n \geq 1$, of harmonic mean converges to x. (**Hint:** Follows from the part a).)

c). Suppose that $x_n > 0$ for all n. Then the sequence $g_n := \sqrt[n]{x_1 \cdots x_n}$ of the geometric mean also converges to x. (**Hint:** Apply Exercise R1.7). By the way by transition to logarithms the assertion follows directly from the part a) because of the continuity of $\ln$ and $\exp$.)

d). With the help of the part c) give solutions to the Exercises R2.2-5) and R2.2-6) and prove that $\lim_{n\to\infty}\sqrt[n]{n!} = \infty$ and $\lim_{n\to\infty}\sqrt[n]{n!}/n = 1/e$. (**Hint:** Both these assertions also follow from Stirling formula, this supply even the sharper assertion $\lim_{n\to\infty} n!/\sqrt{2\pi n}(n/e)^n = 1$.)

e). Show by giving counter examples that the converse assertions in the parts a), b), c) are not true in general.

f). Let (x_n) be a sequence of positive real numbers such that the sequence (x_{n+1}/x_n) converges to x. Then show that the sequence $(\sqrt[n]{x_n})$ also convergers to x.

4). Let (x_n) be convergent (eventually improper) sequence of real numbers with $\lim x_n = x$. Then the sequence $2^{-n}\sum_{m=0}^{n}\binom{n}{m}x_m$, $n \in \mathbb{N}$, is also convergent with limiting value x.

5). Let (x_n) and (y_n) be sequences in $\mathbb{R}$ with $y_n > 0$ and $\lim_{n\to\infty}(y_0 + \cdots + y_n) = \infty$. If the sequence (x_n/y_n) converges to a, then the sequence

$$\Big(\frac{x_0 + \cdots + x_n}{y_0 + \cdots + y_n}\Big)$$

also converges to a .

6). Show that

a). $\lim_{n\to\infty}\big(1-(1/n^2)\big)^n = 1$. (**Hint:** Use the Bernoulli's inequality.)

b). $\lim_{n\to\infty}\big(1-(1/n)\big)^n = 1/e$.

c). $\lim_{n\to\infty}\big(1+(1/n^2)\big)^n = 1$. (**Hint:** Consider $\big(1+(1/n^2)\big)^{n^2}\to e$.)

7). Show that the sequence f_{n+1}/f_n , $n\geq 1$, the quotients of the Fibonacci-sequence converges to $\varphi := (1+\sqrt{5})/2$. (**Remark:** If a segment is divided into the ratio of the golden ratio, i.e. the ratio of the total segment to the bigger part of the segment a which is also the ratio of this bigger part of the segment to the smaller part of the segment b , then $(a+b)/a = a/b = \varphi = 1+\varphi^{-1} = \varphi^2-1$.) [1]) – By the way f_{n+1}/f_n is the $(n-1)$ -th approximation of the continued fraction of $\varphi = [1, 1, 1, \ldots]$.)

R2.4. 1). Discuss the convergence for the following recursively defined sequences (x_n) and compute their limiting values

a). $x_{n+1} = x_n^2 + \frac{1}{4}$, $n\in\mathbb{N}$, with $0\leq x_0\leq \frac{1}{2}$.

b). $x_0 = 0,\ x_{n+1} = \frac{1}{2}(a+x_n^2)$, $n\in\mathbb{N}$, with $0\leq a\leq 1$.

c). $x_0 = 0,\ x_{n+1} = \frac{1}{2}(a-x_n^2)$, $n\in\mathbb{N}$, with $0\leq a\leq 1$.

d). $x_0 = 2,\ x_{n+1} = 2-1/x_n$, $n\in\mathbb{N}$.

e). $x_0 = 0,\ x_{n+1} = \sqrt{a+x_n}$, $n\in\mathbb{N}$, with $a>0$. (**Hint:** For $a=2$, we get the sequence (c_n) of example 4.F.11.)

f). $x_{n+1} = 2x_n - ax_n^2$, $n\in\mathbb{N}$, with $a\in\mathbb{R},\ a>0$ and $0<x_0<2/a$.

g). $x_{n+1} = (x_n+2)/(x_n+1)$ with $x_0\geq 0$.

h). $x_{n+1} = \frac{1}{3}(x_n^2+2)$ with x_0 arbitrary.

(**Remark:** Controll the answers with computer.)

2). a). For the recursively defined sequences (x_n) by $x_0 = a$, $x_1 = b$ and $x_{n+2} = (x_n + x_{n+1})/2$ with $a, b\in\mathbb{R}$, show that $\lim x_n = (a+2b)/3$. (**Hint:** Look at subsequences (x_{2n}) and (x_{2n+1}) separately.)

b). For the recursively defined sequence (x_n) by $x_0 = a,\ x_1 = 1,\ x_{n+2} = \sqrt{x_n x_{n+1}}$ with $a\in\mathbb{R},\ a>0$, show that $\lim x_n = \sqrt[3]{a}$.

3). Let $a\geq 1$. Show that the recursively defined sequence (x_n) with

$$x_0 = a,\ x_{n+1} = a + \frac{1}{x_n},$$

$n\in\mathbb{N}$, is convergent and find its limiting value. (see also the example 4.F.13 on continued fractions.)

4). Let $a, b > 0$. Show that the recursively defined sequences (a_n) and (b_n) with $a_0 = a,\ b_0 = b$,

$$a_{n+1} = \frac{2a_n b_n}{a_n+b_n} = \text{ harmonic mean of } a_n\,,\ b_n \quad \text{and} \quad b_{n+1} = \frac{a_n+b_n}{2} = \text{arithmetic mean of } a_n\,,\ b_n$$

form (after $n=1$) a nested intervals for the geometric mean $\sqrt{ab}$ of a and b . (**Hint:** Note that $a_n b_n = ab$ for all $n\in\mathbb{N}$. The sequence (x_n) in example 4.F.9 show that the sequence (b_n) is obtained if we put $a_0 = a/x_0$ and $b_0 = x_0$. Since $0\leq b_{n+1}-a_{n+1} = (b_n-a_n)^2/2(a_n+b_n)$, we have quadratic convergence of the nested intervals.)

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[1]) φ like $\Phi\varepsilon\iota\delta\iota\alpha\varsigma$. The number φ is often also denoted by τ . — For $\alpha := \pi/5$ from $(4\cos^2\alpha - 1 - 2\cos\alpha)\sin\alpha = \sin 3\alpha - \sin 2\alpha = 0$, we have the equations $4\cos^2\alpha - 2\cos\alpha - 1 = 0$ and $2\cos\alpha = 2\cos(\pi/5) = \varphi$. *Therefore* $\cos(\pi/5)$ *and hence the regular* 10*-gon* (*as well as the regular* 5*-gon*) *with the golden ratio can be constructed.* See the representation of ζ_5 in ?????.

5). Let $a, b > 0$. Show that the recursively defined sequences (a_n) and (b_n) with $a_0 = a,\ b_0 = b$,

$$a_{n+1} = \frac{a_n + b_n}{2} = \text{arithmetic mean of } a_n\,,\ b_n \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n} = \text{geometric mean of } a_n\,,\ b_n$$

form (after $n = 1$) a nested intervals $[b_n\,, a_n]$, $n \in \mathbb{N}^*$. (**Remark:** the number $M(a, b)$ defined by this nested intervals is called the arithmetic-geometric mean of a and b . Since

$$0 \le a_{n+1} - b_{n+1} = (a_n - b_n)^2/2\left(a_n + b_n + 2\sqrt{a_n b_n}\right)$$

here also we have quadratic convergence of the nested intervals. — Similarly, by using the harmonic and geometric mean we can define the nested intervals and the number $1\big/M\big(1/a\,, 1/b\big)$.)

6). Prove that the recursively defined sequence (x_n) defined at the end of example 4.F.9 converges to $\sqrt[k]{a}$, where for $n \ge 1$ the we have the following error :

$$0 \le x_n - \sqrt[k]{a} \le \frac{1}{k(\sqrt[k]{a})^{k-1}}\,(x_n^k - a)\,.$$

7). Let $a \in \mathbb{R}_+^\times$. The recursively defined sequence (x_n) with $x_0 > 0$ arbitrary and

$$x_{n+1} = \frac{x_n^2 + 3a}{3x_n^2 + a}\,x_n$$

converges monotonically to $\sqrt{a}$, where since $x_{n+1} - \sqrt{a} = (x_n - \sqrt{a})^3/(3x_n^2 + a)$, we have even cubic convergence.

8). Let (a_n) be a monotone increasing and let (b_n) be a montonoe decreasing sequences of real numbers with $a_n \le b_n$ for all $n \in \mathbb{N}$ and $a := \lim a_n$, $b := \lim b_n$. Show that $\bigcap_{n=0}^{\infty}[a_n\,, b_n] = [a\,, b]$.

R2.5. 1). a). Compute the dual and ternary expansions of $1/7,\ 1/8,\ 1/9,\ 1/10$.

b). Compute the g-adic expansions of $a/(g-1)$ and $a/(g+1)$. Further, prove $1/(g-1)^2 = (0, \overline{012\ldots g-3\ g-1})_g$, where the overlined sequence of digits repeats iteself periodically.

2). In every interval of R (with more than one point) there are infinitely many rational numbers and there are infinitely many irrational numbers. Deduce that the set of irrational numbers is dense in R .

3). For $x \in \mathbb{R}$, the sequence of rational numbers $[nx]/n,\ n \ge 1$, converges to x . (**Remark:** See also example 4.F.12.)

4). The product representation $2/\pi = \prod_{n=1}^{\infty} c_n/2$ of Vieta (see example 4.F.11) can be interpreted as follows : A square with the side length 1 is circumscribed in a circle with the diameter $d_1 (= \sqrt{2})$, in this regular 8 -gon is again circumscribed in a circle with the diameter d_2 , in this regular 16 -gon is again circumscribed etc.. Then the sequence of diameters (d_n) converges to $\pi/2$.

5). For $a, b \in \mathbb{N}^*$, show that $[a, b, a, b, a, b, \ldots] = (ab + \sqrt{a^2b^2 + 4ab})/2b$ (**Remark:** see also the example 4.F.13) .)

6). For $n \in \mathbb{N}^*$, show that $\sqrt{n^2+1} = [n, 2n, 2n, 2n, \ldots]$, $\sqrt{n^2+2} = [n, n, 2n, n, 2n, \ldots]$, $\sqrt{(n+1)^2 - 1} = [n, 1, 2n, 1, 2n, \ldots]$ (**Remark:** See also Example 4.F.13) .)

7). Let m_k and n_k , $k \in \mathbb{N}^*$, be sequences of positive natural numbers with the following properties : (1) For all k , $m_k \le n_k$. (2) $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$. (3) The limiting value $x := \lim_{k\to\infty}(n_k/m_k)$ exists. (For example: $m_k := k,\ n_k := [kx]$ with $x \ge 1$ fixed, see exercise 3) above.) Then

$$\lim_{k\to\infty} \sum_{n=m_k+1}^{n_k} \frac{1}{n} = \ln x\,.$$

In particular, ($m_k := k,\ n_k := 2k$)

$$\ln 2 = \lim_{k\to\infty} \sum_{n=k+1}^{2k} \frac{1}{n} = \lim_{k\to\infty} \sum_{n=1}^{2k} (-1)^{n-1}\frac{1}{n} = \lim_{k\to\infty} \sum_{n=1}^{k} (-1)^{n-1}\frac{1}{n} =: \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}\,.$$

(**Hint:** Example 4.F.10. — Use the continuity of the Logarithm: If the sequence $x_k \in \mathbb{R}_+^\times,\ k \in \mathbb{N}$ converges to $x \in \mathbb{R}_+^\times$, then the sequence $\ln x_k$, $k \in \mathbb{N}$, converges to $\ln x$.)

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† **Constantin Carathéodory (1873-1950)** Constantin Carathéodory was born on 13 Sept 1873 in Berlin, Germany and died on 2 Feb 1950 in Munich, Germany. Constantin Carathéodory's father, Stephanos Carathéodory, was an Ottoman Greek who had studied law in Berlin and then served as secretary in the Ottoman embassies in Berlin, Stockholm and Vienna. Stephanos had married Despina Petrocochino, who came from a Greek family of businessmen who had settled in Marseille. At the time of Constantin's birth, the family were in Berlin since Stephanos had been appointed there two years earlier as First Secretary to the Ottoman Legation.

The Carathéodory family spent 1874-75 in Constantinople, where Constantin's paternal grandfather lived, while Stephanos was on leave. Then in 1875 they went to Brussels when Stephanos was appointed there as Ottoman Ambassador. In Brussels, Constantin's younger sister Loulia was born. The year 1895 was a tragic one for the family since Constantin's paternal grandfather died in that year, but much more tragically, Constantin's mother Despina died of pneumonia in Cannes. Constantin's maternal grandmother took on the task of bringing up Constantin and Loulia in his father's home in Belgium. They employed a German maid who taught the children to speak German. Constantin was already bilingual in French and Greek by this time. Constantin began his formal schooling at a private school in Vanderstock in 1881. He left after two years and then spent time with his father on a visit to Berlin, and also spent the winters of 1883-84 and 1884-85 on the Italian Riviera. Back in Brussels in 1885 he attended a grammar school for a year where he first began to become interested in mathematics. In 1886 he entered the high school Athénée Royal d'Ixelles and studied there until his graduation in 1891. Twice during his time at this school Constantin won a prize as the best mathematics student in Belgium.

At this stage Carathéodory began training as a military engineer. He attended the École Militaire de Belgique from October 1891 to May 1895 and he also studied at the É d'Application from 1893 to 1896. In 1897 a war broke out between Turkey and Greece. This put Carathéodory in a difficult position since he sided with the Greeks, yet his father served the government of the Ottoman Empire. Now a trained engineer was offered a job in the British colonial service. This job took him to Egypt where he worked on the construction of the Assiut dam until April 1900. During periods when construction work had to stop due to floods, he studied mathematics from some textbooks he had with him such as Jordan's "Cours d'Analyse" and Salmon's text on the "analytic geometry of conic sections". He also visited the Cheops pyramid and made measurements which he wrote up and published in 1901. He also published a book "Egypt" in the same year which contained a wealth of information on the history and geography of the country.

Carathéodory entered the University of Berlin in May 1900 where Frobenius and Schwarz were professors. He attended Frobenius's lectures but benefited most from a twice monthly colloquium run by Schwarz who was lecturing on his collected works. He also became close friends with Fejér while at Berlin. After hearing excellent reports of mathematics research at Göttingen, he decided to continue his studies there and enrolled for the summer semester of 1902. Carathéodory was indeed impressed with Göttingen, describing it as the : ... *seat of an international congress of mathematicians permanently in session.*

He worked on the calculus of variations and was much influenced by both Hilbert and Klein. He received his doctorate in 1904 from Göttingen University for his thesis "Über die diskontinuierlichen Lösungen in der Variationsrechnung" which he submitted to Hermann Minkowski. His oral examination was held on 13 July when he was also examined in his subsidiary subjects of applied mathematics and astronomy by Klein and Schwarzschild. He remained at Göttingen to write his habilitation thesis "Über die starken Maxima und Minima bei einfachen Integralen" which he submitted on 5 March 1905. He then lectured as a Privatdozent at Göttingen until 1908.

Carathéodory had spent time in Brussels with his father Stephanos over the summer of 1907. After a few months of deteriorating health Stephanos died in late 1907. Study, at Bonn, had proposed Carathéodory as Furtwängler's successor and, after serious thought as to whether he should leave Göttingen, Carathéodory went to Bonn where he became a Provatdozent on 1 April 1908. At Bonn he collaborated with Study on isoperimetric problems. On 5 February 1909 he married Euphrosyne Carathéodory in Constantinople. In marrying Euphrosyne, who was his aunt and eleven years his junior, Carathéodory was following a family tradition of marrying close relatives. After a year at Bonn, Carathéodory was appointed as Professor of higher Mathematics at the Technical University of Hanover, so becoming Stäckel's successor. Again it was not long before he moved on and on 1 October 1910 he was appointed to the Chair of Higher Mathematics at the Technical University of Breslau. This time he held the chair for two and a half years before being appointed professor at Göttingen from 1 April 1913. The years of World War I were difficult ones for Carathéodory and his family. Most of his colleagues and students served in the military and he was isolated in Göttingen. The famine of 1917 hit hard but Carathéodory continued to give lecture courses at the university.

After five years Göttingen he was appointed to the University of Berlin in 1918 but after he had been there for a year, at the request of the Greek government, he ended his contract with Berlin on 31 December 1919 and travelled to Greece to undertake a new venture. By this time Constantin and Euphrosyne had two children, Stephanos born in Hanover on 7 November 1909 and Despina born on 13 October 1912. Carathéodory had also accepted editorial positions on the boards of two major mathematics journals, the “Rendiconti del Circolo Matematico di Palermo” from 1909 and the “Mathematische Annalen” from 1914. The Greek government had asked Carathéodory to establish a second university in Smyrna. However, he also required a university post so he was appointed as Professor of Analytical and Higher Geometry at the University of Athens on 2 June 1920. On 14 July the Greek government published a bill setting up a Greek University in Smyrna and soon others were appointed at assist Carathéodory. On 28 July Carathéodory was officially appointed as organiser of the Ionian University in Smyrna and also Professor of Mathematics at the new university. In the second half of 1921 he travelled widely through Europe purchasing books and equipment for the new university. The Turks attacked Smyrna in September 1922 and so the planned opening of the university in October of that year became impossible. Carathéodory was able to save the university library, which he had worked so hard to establish, and most of the equipment which he had been purchased for the science departments, and escaped to Athens on a Greek battleship. He taught at Athens at the National University and the National Technical University until 1924 when he moved to Munich to fill the chair left vacant when Lindemann retired.

In 1928 Carathéodory became the first visiting lecturer of the American Mathematical Society. He sailed to the United States with his wife in January and after a lecture tour and time spent as a visiting professor at Harvard, returned to Munich in September. In the following year he received an offer of a post from Stanford university and was in fact appointed there in September 1929. However, he seems to have only been using this offer as a means of getting better salary and conditions from Munich, which indeed he managed to do. On 30 January 1933 the National Socialist party led by Hitler came to power in Germany.

Carathéodory could hardly conceive how this could happen in a country with the cultural traditions of Germany. He initially tended to view the Hitler regime with a somewhat overconfident contempt, whereas later, when Hitler gained absolute power, he was incapable of resistance. His behaviour in the Nazi era was, in fact, identical with that of the ... educated bourgeois who, despite their humanistic background, in their overwhelming majority abstained from any opposition against Hitler’s dictatorship, and especially Hitler’s war, and thus dramatically failed to exercise their historic responsibility towards both Germany and humanity as a whole.

Carathéodory continued to hold his position in Munich until he retired in August 1938. However he certainly undertook many duties which took him to other places. In particular he continued to work on reorganising the Greek universities, particularly during 1930-32, with the aim of integrating Greece academically into Europe. In 1936-37 he made another visit to the United States, giving a lecture at the American Mathematical Society meeting to commemorate the tercentenary of Harvard University on 31 August 1936, then spending the winter semester at the University of Wisconsin as Carl Schurz Memorial Professor.

The World War II was a difficult time for Carathéodory. Georgiadou writes :

... during World War II he took part in the procedures of the Bavarian Academy of Sciences. He did not get involved in the movement for national socialism, but he did have connections with Nazi party members [particularly Hasse, Blaschke and Süss]. *He never openly mentioned the holocaust or the Nazi crimes against Greece. ... kept silent in the face of crimes that violated any idea of human decency, accepted the authority of an illegal state, made his compromises and submitted to the expulsion of Jews from scientific institutions ... However, he took great pains to re-establish mathematics as an academic discipline in Germany after the war and thus to contribute to the reintegration of this country into the community of civilised nations.*

Carathéodory made significant contributions to the calculus of variations, the theory of point set measure, and the theory of functions of a real variable. He added important results to the relationship between first order partial differential equations and the calculus of variations. He contributed important results to the theory of functions of several variables. He examined conformal representations of simply connected regions and he developed a theory of boundary correspondence. He also made contributions in thermodynamics, the special theory of relativity, mechanics, and geometrical optics.

Carathéodory wrote many fine books including ‘Lectures on Real Functions (1918)”, “Conformal representation (1932)”, “Calculus of Variations” and “Partial Differential Equations (1935)”, “Geometric Optics (1937)”, “Real functions Vol. 1: Numbers, Point sets, Functions (1939)”, and “Funktionentheorie”, a 2 volume work published in 1950.

One might wonder why there is no “Real functions Vol. 2” in this list. In fact Carathéodory did write the second volume of this work but it was destroyed while at the publisher Teubner during the bombing of Leipzig in 1943.

Perron, writing in 1952, remarks that Carathéodory : ... *had not published many of his ideas; they result in others works, especially in those of the numerous students who were introduced by him to the spirit and ways of scientific research and who partly themselves occupy university chairs today.*

He supervised two doctoral students at Göttingen (Hans Rademacher and Paul Finsler), one at Berlin, and 17 at Munich.