MA-302 Advanced Calculus

1. Differentiable curves and functions

1.1. a). Let A be a finite dimensional $\mathbb{K}$- algebra and let $g : I \to A$ be a differentiable curve in A such that the image of g is contained in the unit group $A^\times$ of A [1]). Show that the curve $g^{-1} : I \to A$ defined by $g^{-1}(t) := (g(t))^{-1}$ is differentiable and the derivative of g^{1-} is $\left(g^{-1}\right)' = -g^{-1}g'g^{-1}$. Moreover, for any differentiable curve $f : I \to A$, prove the quotient rules:

$$\left(fg^{-1}\right)' = \left(f' - fg^{-1}g'\right)g^{-1} \quad \text{and} \quad \left(g^{-1}f\right)' = g^{-1}\left(f' - g'g^{-1}f\right).$$

b). Let Δ be a determinant function on the n- dimensional $\mathbb{K}$-vector space V and let $f_j : I \to V$ be differentiable curves in V, $j = 1, \ldots, n$. Show that the function $\Delta(f_1, \ldots, f_n) : I \to \mathbb{K}$, is differentiable with $\Delta(f_1, \ldots, f_n)' = \sum_{j=1}^n \Delta(f_1, \ldots, f_j', \ldots f_n)$.

1.2. Let $f : I \to V$ be a differentiable curve in the Euclidean vector space V. If f is differentiable at $t_0 \in I$ and $f(t_0) \neq 0$, then the curve $||f|| : I \to \mathbb{R}$, $t \mapsto ||f(t)||$, is also differentiable at t_0 and

$$\|f\|'(t_0) = \frac{\langle f'(t_0), f(t_0)\rangle}{\|f(t_0)\|}.$$

1.3. Let $f : I \to V$ be a k-times differentiable curve in a finite dimensional $\mathbb{K}$-vector space V. Suppose that $f^{(k)}(t) = a_{k-1}(t)f^{(k-1)}(t) + \cdots + a_1(t)f'(t)$, where $a_1, \ldots, a_{k-1} : I \to \mathbb{K}$ are continuous functions. Then the trajectory of f is contained in the atmost $(k-1)$-dimensional affine subspace $\sum_{i=1}^{k-1} \mathbb{K}f^{(i)}(t_0) + f(t_0)$, where $t_0 \in I$ is fixed (but arbitrary).

1.4. Let g be a continuous real-valued function on the unit circle $\mathrm{S}^1 := \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$ such that $g((0,1)) = g((1,0)) = 0$ and $g(-x) = -g(x)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} ||x|| \cdot g\left(\dfrac{x}{||x||}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

a). For a fixed $x \in \mathbb{R}^2$, show that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) := f(tx)$ is differentiable.

b). Show that f is not differentiable at $(0,0)$ unless $g = 0$.

1.5. a). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := \sqrt{|xy|}$. Show that f is not differentiable at 0.

b). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq ||x||^2$ for all $x \in \mathbb{R}^n$. Show that f is differentiable at 0.

[1]) For example, $A = \mathrm{M}_n(\mathbb{K})$ or $\mathrm{End}_{\mathbb{K}}(V)$, V finite dimensional $\mathbb{K}$-vector space ; $A^\times = \mathrm{Gl}_n(\mathbb{K})$ or $\mathrm{Aut}_{\mathbb{K}}(V)$.