MA-302 Advanced Calculus

3. Directional Derivatives

3.1. a). Compute $\dfrac{\partial^{102}}{\partial y^2 \partial x^{100}}\big((1+x^2)^x y\big)$.

b). Which directional derivatives and which partial derivatives exist for the following functions $f:\mathbb{R}^2 \to \mathbb{R}$ (at the origin all these functions are defined to be $f(0,0) := 0$):

(i) $f(x, y) := xy/(x^2+y^2)$. (ii) $f(x, y) := x^3/(x^2+y^2)$. (iii) $f(x, y) := x^2y/(x^2+y^2)$.

(iv) $f(x, y) := x^3/(x^4+y^2)$. (v) $f(x, y) := x^2y/(x^4+y^2)$.

3.2. Let $G \subseteq \mathbb{R}^2$ and $f : G \to \mathbb{R}$ be partially differentiable function. If $\mathrm{D}_2 f$ is continuous at the point $x_0 \in G$, then show that f is differentaible in all directions at x_0.

3.3. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a function.

a). Suppose that the partial derivative $\partial f/\partial x$ exists and is identically 0. Then show that there exsits a function $g:\mathbb{R}\to\mathbb{R}$ such that $f(x, y) = g(y)$ for all $x, y \in \mathbb{R}$.

b). Suppose that the second partial derivative $\partial^2 f/\partial x \partial y$ exists and is identically 0. Then show that there exist functions $g, h:\mathbb{R}\to\mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for all $x, y \in \mathbb{R}$.

3.4. Let $G \subseteq V$ be a domain (open connected subset) and let $f : G \to W$ be partially differentiable with respect to a basis of V. Suppose that all partial derivatives of f are identically 0, then show that f is a constant function. (**Hint:** Reduce to the case $\mathbb{K} = \mathbb{R}$.)

3.5. Let $v_1, \ldots, v_n$ be a basis of V. Suppose that for the map $f : G \to W$ all the partial derivatives $\mathrm{D}_{v_i} f$, $i = 1, \ldots, n$, exist on G and are bounded in a neighbourhood of $x_0 \in G$. Show that f is continuous at x_0.