MA-302 Advanced Calculus

6. Power series, Analytic functions

Karl Theodor Wilhelm Weierstrass †
(1815-1897)

The exercises 6.1 and 6.5 are only to get practice, their solutions need not be submitted.

6.1. Find the domain of convergence D_F of the following power series $F_1, F_2, F_3, F_4 \in \mathbb{R}[\![X_1, X_2]\!]$ and $F_5, F_6 \in \mathbb{C}[\![X_1, X_2]\!]$:

a). $F_1 := \sum_{m\in\mathbb{N}^2} \binom{m_1+m_2}{m_1} X^m = \sum_{n=0}^{\infty} (X_1 + X_2)^n$. **b).** $F_2 := \sum_{m\in\mathbb{N}^2} m_1^{m_2} X^m$.

c). $F_3 := \sum_{m\in\mathbb{N}^2} \frac{m_1^{m_2}}{m_2!} X^m$. **d).** $F_4 := \sum_{m\in\mathbb{N}^2} \frac{m_1^{m_1}}{m_2!} X^m$.

e). $F_5 := \sum_{m\in\mathbb{N}^2} \frac{2^{m_1}}{m_2!} X^m$. **f).** $F_6 := \sum_{m\in\mathbb{N}^2} \delta_{m_1 m_2} X^m$, where δ_{ij} denote the Kronecker's delta.

6.2. Let $F_1 = \sum_{k\in\mathbb{N}} a_k Y^k$, $F_2 = \sum_{k\in\mathbb{N}} b_k Y^k \in \mathbb{K}[Y]$, be non-zero power series in one indeterminate Y over $\mathbb{K}$ and let $F = \sum_{m\in\mathbb{N}^2} a_{m_1} b_{m_2} X_1^{m_1} X_2^{m_2} \in \mathbb{K}[X_1, X_2]$. Show that $\mathrm{D}_F = \mathrm{D}_{F_1} \times \mathrm{D}_{F_2}$. Moreover, if F is convergent, then show that $F(x) = F_1(x_1)\cdot F_2(x_2)$ for every $x = (x_1, x_2) \in \mathrm{D}_F$.

6.3. For the vectors $c_m \in W$, $m \in \mathbb{N}^n$, and $t \in (\mathbb{R}_+^\times)^n$, suppose that the family $c_m t^m$, $m \in \mathbb{N}^n$, is bounded. Then show that $\prod_{i=1}^n \mathrm{B}(0\,;t_i) \subseteq \mathrm{D}_F$ the domain of convergence of the power series $\sum_{m\in\mathbb{N}^n} c_m X^m \in W[\![X_1, \ldots, X_n]\!]$.

6.4. Let $t \in (\mathbb{R}_+^\times)^n$. Show that $\mathrm{B}_t(W) := \{F \in W\langle\!\langle X_1, \ldots, X_n\rangle\!\rangle \mid \|F\|_t < \infty\}$ is a Banach space over $\mathbb{K}$ with respect to the t-norm. Moreover, if $W = \mathbb{K}$, then $\mathrm{B}_t := \mathrm{B}_t(\mathbb{K})$ is a $\mathbb{K}$-Banach algebra.

6.5. Show that the following functions f are analytic at given x_0 and find its power series expansion at x_0.

a). $f(x, y) = e^{x+y}$ at $x_0 = (a, b) \in \mathbb{K}^2$. **b).** $f(x, y) = e^{xy} \sin y$ at $x_0 = (0, 0)$.

c). $f(x, y) = \sin(x+y/2)$ at $x_0 = (a, b) \in \mathbb{K}^2$. **d).** $f(x, y) = 1/(1-x-xy)$ at $x_0 = (0, 0)$.

e). $f(x, y) = (4x+y^2)\exp(-x^2-4y^2)$ at $x_0 = (0, 0)$.

f). $f(x, y) = x^y$ at $x_0 = (1, 0)$. (**Hint:** $x^y = \sum_{n,k} (-1)^{n+k} \dfrac{s(n, k)}{k!} (x-1)^k y^n$, where $s(n, k)$ are the Stirling's numbers of first kind. For $0 \le n \le m$, they are defined by the equation: $\binom{x}{m} = \frac{1}{m!} \sum_{n=0}^{m} (-1)^{m-n} s(m, n) x^n$, and $s(m, n) = 0$ otherwise.)

g). $f(x, t) = xe^{xt}/(e^x - 1)$ at $x_0 = (0, 0)$. (**Hint:** $\frac{xe^{xt}}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n$, where $B_n(t) \in \mathbb{Q}[t]$ are the Bernoulli polynomials. They are defined by the equation: $\frac{Xe^{TX}}{e^X-1} = \sum_{n=0}^{\infty} \frac{B_n(T)}{n!} X^n \in \mathbb{Q}[T][\![X]\!]$.)

6.6. Let V, W be finite dimensional complex vector spaces and $G \subseteq V$ be an open subset. A map $G \to W$ is complex analytic if and only if it is real analytic and it satisfies the Cauchy-Riemann differential equations.

6.7. Let $F_1, F_2 : G \to \mathbb{K}$ be analytic functions defined on an open subset $G \subseteq \mathbb{K}^m$. If $F_2 \neq 0$, then show that F_1/F_2 is analytic on the open subset $\{x \in G \mid F_2(x) \neq 0\}$. In particular, polynomial functions and rational functions are analytic functions.

6.8. a). Give an example of a non-zero complex analytic function $F : \mathbb{C}^2 \to \mathbb{C}$ and a sequence $z_n \in \mathbb{C}^2$, $n \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \in \mathbb{N}$, $z_n \to 0$ and $F(z_n) = 0$ for every $n \in \mathbb{N}$.

b). Give an example of a complex analytic map $F : \mathbb{C}^2 \to \mathbb{C}^2$ which is neither open nor constant.

c). Show that there is no analytic function $F : \mathbb{K}^m \to \mathbb{K}$ such that $F(0) = 1$ and $F(x) = 0$ for all $x \in K^m$ with $|x| > 1$.

† **Karl Theodor Wilhelm Weierstrass (1815-1897)** was born on Oct 1815 in Ostenfelde, Westphalia (now Germany) and died on 19 Feb 1897 in Berlin, Germany. While at the Gymnasium Weierstrass certainly reached a level of mathematical competence far beyond what would have been expected. He regularly read Crelle's Journal and gave mathematical tuition to one of his brothers. However Weierstrass's father wished him to study finance and so, after graduating from the Gymnasium in 1834, he entered the University of Bonn with a course planned out for him which included the study of law, finance and economics. However, Weierstrass suffered from the conflict of either obeying his father's wishes or studying the subject he loved, namely mathematics. The result of the conflict which went on inside Weierstrass was that he did not attend either the mathematics lectures or the lectures of his planned course. He reacted to the conflict inside him by pretending that he did not care about his studies, and he spent four years of intensive fencing and drinking.

He did study mathematics on his own, however, reading Laplace's Méchanique céleste and then a work by Jacobi on elliptic functions. He came to understand the necessary methods in elliptic function theory by studying transcripts of lectures by Gudermann. In a letter to Lie, written nearly 50 years later, he explained how he came to make the definite decision to study mathematics despite his father's wishes around this time.

... when I became aware of [a letter from Abel to Legendre] in Crelle's Journal during my student years, [it] was of the utmost importance. The immediate derivation of the form of the representation of the function given by Abel ..., from the differential equation defining this function, was the first mathematical task I set myself; and its fortunate solution made me determined to devote myself wholly to mathematics; I made this decision in my seventh semester ...

Now Weierstrass had made a decision to become a mathematician but he was still supposed to be on a course studying public finance and administration. After his decision, he spent one further semester at the University of Bonn, his eighth semester ending in 1838, and having failed to study the subjects he was enrolled for he simply left the University without taking the examinations. Weierstrass's father was desperately upset by his son giving up his studies. He was persuaded by a family friend, the president of the law courts at Paderborn, to allow Karl to study at the Theological and Philosophical Academy of Münster so that he could take the necessary examinations to become a secondary school teacher.

On 22 May 1839 Weierstrass enrolled at the Academy in Münster. Gudermann lectured in Münster and was the reason that Weierstrass was so keen to study there. Weierstrass attended Gudermann's lectures on elliptic functions, some of the first lectures on this topic to be given, and Gudermann strongly encouraged Weierstrass in his mathematical studies. Leaving Münster in the autumn of 1839, Weierstrass studied for the teacher's examination which he registered for in March 1840. |sevensc Gudderman Weierstrass's teacher, in his evaluation wrote: "*With this work the candidate enters the ranks of famous inventors as co-equal*." Gudderman urged publication of the exam project as soon as possible and that would have happened had the philosophy faculty of the royal academy at Münster/Westphalia at that time had the authority to grant degrees. "*Then we would have the pleasure of counting Weierstrass amoung our doctoral graduates*", so was written in 1887 rector's address of Weierstrass' formal pupil W. Killing (whose name was later immortalized in Lie theory). Not until 1894, fifty-four years after it was written, did Weierstrass publish his exam work.

1842-1848 teacher at the Progymnasium in Deutsch-Krone, West Prussia, of mathematics, penmanship and gymnastics; 1848-1855 teacher at the Gymnasium in Braunsberg, East Prissia;1854 publication of trail-blazing results (gotten already in 1849) "Zur Theorie der Abelschen Functionen" in Journal für Reine und Angew. Math., thereupon honorary doctorate from the University of Königberg and promotion to assistant headmaster; 1856 at the instigation of A. Von Humbolt and L. Crele appointment as professor at the Industrial Institute (later Techinal University) at Berlin; 1857 adjunct professor at the University of Berlin; after 1860 lectures often with more than 200 auditors; 1861 breakdown from over work; 1864 at the age of almost fifty appointmnet to an ordinary professorship, created for him, at the University of Berlin; 1873/74 rector magnificus there, member of numerous academies at home and abroad;1885 stamping of Weierstrass medal (for his 70th birthday); 1890 teaching activity halted by serious illness, confinement to wheelchair; 1895 festive unveilling his image in the national gallery (80th birthday); 1897 died in Berlin.

Unfortunately, unlike Cauchy, Weierstrass never wrote his lectures out in the book form , but there are transcriptions by his various puplis, for example, from H. A. Schwarz's hand there is an elaboration of his lectures on *Differentialrechnung* held at the Royal Industrial Institute in the 1861 summer semester. There is also a transcription by A. Hurwitz of his summer semester 1878 lectures on *Einleitung in die theorie der analytischen Funktionen* and another by W. Killing. Weierstrass's lectures became world famous; when in 1873 –two years after the Franko-Prussian War –Mittag-Leffler came to Paris to study, Hermite told him: "*you have made a mistake, sir; you should have attended Weierstrass' course in Berlin. He is the master of us all*".

There is no exhaustive biography of Weierstrass, but in the personal remarks made by A. Kneser describes the mathematical life in the 1880's thus: *The undisputed master of the whole opeartion was without doubt Weierstrass, a regal and in every way imposing figure. All knew the magnificent white-locked head, the shining blue eyes slightly dropping at the corners which belonged to the country boy of pure Westphalian stock. By this time his lectures had evolved to a high level of perfection in presentation as well as content and only seldom were those tense minutes experienced where the great man flatered and even the promptings of his faithful assistant at the balckboard, perhaps my friend Richard Müller, couldn't get him back on the track; then he would sink into majestic silence for few minutes; two hundred pairs of young eyes were riveted on the splendid brow with the devout conviction that behind that shining facade the greatest intellect was at work. There were in fact, two hundred youths who attended and listened intently to Weierstrass's lectures on elliptic functions, fully aware that at that time such things never came up on any state examination, a dazzling testimonial to intellectual sprit of times. People even knew very little about the applications of these things, although there were already available some very beautiful ones. The doctrine of the primacy of applied mathematics, of the greater worth of applications as against pure mathematics, had not yet been discovered. The humor of the young was unleashed even on this great man: he was considered a connoisseur of wine and the Berliners, who mocked his westphalian pronunciation, claimed to have actually heard from him the following quintessential example: I'd kladly kulp a kood klass of Burkundy.* – the k's here should be read as g's.

Weierstrass by his lectures in Berlin, influenced mathematics in Germany like no one else. The assistant headmaster from East prussia became the "praeceptor mathematicus Germaniae".