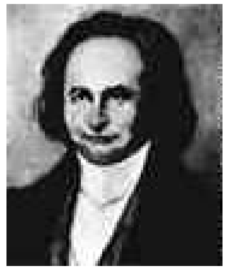
MA-302 Advanced Calculus

8. Inverse function theorem



Carl Gustav Jacob Jacobi[†]
(1804-1851)

The exercises 8.1 and 8.2 are only to get practice, their solutions need not be submitted.

8.1. Determine at which points a of the domain of definition G the following maps F have differentiable inverse and give the biggest possible open subset on which F define a diffeomorphism.

a). $F:\mathbb{R}^2\to\mathbb{R}^2$ with $F(x,y):=(x^2-y^2, 2xy)$. (In the complex case this is the function $z\mapsto z^2$.)

b). $F:\mathbb{R}^2\to\mathbb{R}^2$ with $F(x,y):=(\sin x\cosh y, \cos x\sinh y)$. (In the complex case this is the function $z\mapsto \sin z$.)

c). $F:\mathbb{R}^2\to\mathbb{R}^2$ with $F(x,y):=(x+y, x^2+y^2)$.

d). $F:\mathbb{R}^3\to\mathbb{R}^3$ with $F(r,\varphi,h):=(r\cos\varphi, r\sin\varphi, h)$. (cylindrical coordinates)

e). $F:(\mathbb{R}_+^\times)^2\to\mathbb{R}^2$ with $F(x,y):=(x^3/y, y^3/x)$.

f). $F:\mathbb{R}^2\to\mathbb{R}^2$ with $F(x,y):=(x^2+y^2, e^{xy})$.

8.2. Show that the following maps F are locally invertible at the given point a, and find the Taylor-expansion of the inverse map at the point $F(a)$ upto order 2.

a). $F:\mathbb{R}^3\to\mathbb{R}^3$ with $F(x,y,z):=\big(-(x+y+z), xy+xz+yz, -xyz\big)$ at $a=(0,1,2)$ resp. $a=(-1,2,1)$. (**Hint:** The problem consists in approximating the three zeros of the monic polynomial of degree 3 whose coefficients are close to those of $X(X-1)(X-2)$ resp. $(X+1)(X-2)(X-1)$. – In the general case the map $\Phi_n:\mathbb{K}^n\to\mathbb{K}^n$, which maps the point $(x_1,\dots,x_n)\in\mathbb{K}^n$ to the coefficient tuple $(-s_1, s_2, \dots, (-1)^n s_n)$ of the monic polynomial $(X-x_1)\cdots(X-x_n)=X^n-s_1X^{n-1}+\cdots+(-1)^n s_n$ which has the zeros $x_1,\dots,x_n$, is locally invertible precisely at the points $a=(a_1,\dots,a_n)\in\mathbb{K}^n$ which has distinct components. Upto a sign the Jacobian determinant $(-1)^{\binom{n+1}{2}}\prod_{1\le i<j\le n}(x_i-x_j)=(-1)^n\,\mathrm{V}(x_1,\dots,x_n)$ of Φ_n is a Vandermonde's determinant. To find the zeros of polynomials of degree n, one need to invert the map Φ_n.)

b). $F:\mathbb{R}^3\to\mathbb{R}^3$ with $F(x,y,z):=(x^2+y^3z, e^{xy}, \sin x+\sin y)$ at $a=(0,\pi/2,0)$.

c). $F:\mathbb{R}^2\to\mathbb{R}^2$ with $F(x,y)=(e^{x+y}, xe^y)$ at $a=(0,0)$.

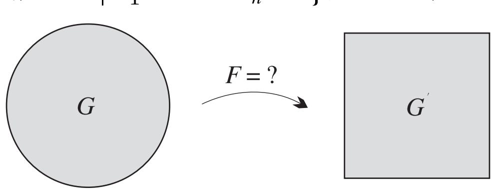
8.3. Let $G\subseteq V$ be an open subset and let $F:G\to V$ be a continuously differentiable map with a fixed point $a\in G$. If 1 is not an eigenvalue of the total differential $(\mathrm{D}F)_a:V\to V$ of F at the point a, then show that a is an isolated fixed point of F (i.e. there exists an open neighbourhood of a, in which F has no other fixed point).

8.4. Find a (real) C^ω-diffeomorphism $F:G\to G'$ for the following open subsets $G\subseteq V$, $G'\subseteq W$.

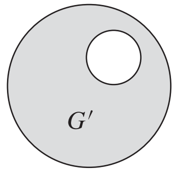
a). $G:=\mathbb{R}^n$, $G':=\mathrm{B}(x_0\,;r)$, $r>0$ and W is an inner product space with $\mathrm{Dim}_{\mathbb{R}}W=n$.

b). $G:=\mathbb{R}^n$, $G':=(a_1,b_1)\times\cdots\times(a_n,b_n)\subseteq\mathbb{R}^n$, $-\infty\le a_i<b_i\le\infty$, $i=1,\dots,n$.

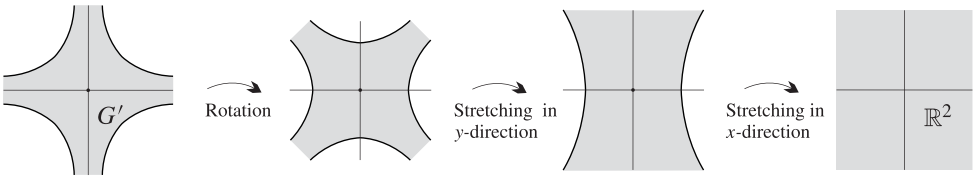
c). $G:=\big\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\big|\; x_1^2+\cdots+x_n^2<1\big\}$, $G':=(-1,1)^n\subseteq\mathbb{R}^n$.



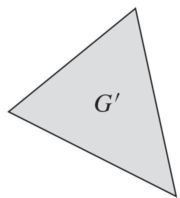
d). $G := \mathbb{R}^n - \{0\}$, $G' := \mathrm{B}(x_0\,;r) - \overline{\mathrm{B}}(y_0\,;s)$, $0 \le s \le s + \|y_0 - x_0\| < r$ and W is an inner product space with $\mathrm{Dim}_{\mathbb{R}} W = n$.



e). $G := \mathbb{R}^2$, $G' := \big\{(x, y) \in \mathbb{R}^2 \bigm| |xy| < 1\big\}$.



f). $G := \mathbb{R}^n$, G' is the interior of a non-degenerate n-simplex in W, where $n := \mathrm{Dim}_{\mathbb{R}} W$. (Induction on n.)

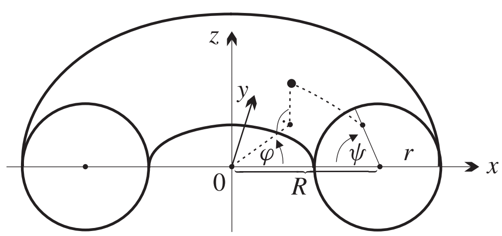


8.5. (Cycloide-coordinates) The map $(a, \sigma) \mapsto a(\sigma - \sin\sigma\,, 1 - \cos\sigma)$ is a C^ω-diffeomorphism of $\mathbb{R}_+^\times \times (0, 2\pi)$ onto $\mathbb{R}_+^\times \times \mathbb{R}_+^\times$. (The coordinate-lines $a = a_0 =$ const. are cycloids.)

8.6. (Torus-coordinates) Let $R > 0$. The map

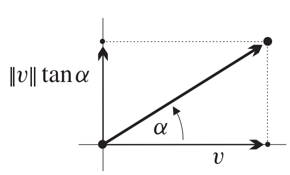
$$T : (r, \varphi, \psi) \mapsto \big((R - r\cos\psi)\cos\varphi\,, (R - r\cos\psi)\sin\varphi\,, r\sin\psi\big)$$

is a C^ω-diffeomorphism of $(0, R) \times (0, 2\pi)^2$ onto the slit open torus.



The coordinate-system (r, φ, ψ) is orthogonal. Compute the Jacobian determinant of T and give the Laplace-Operator Δ in the coordinates r, φ, ψ.

8.7. (Cone-coordinates) Let V be an Euclidean vector space. The map $(v, \alpha) \mapsto (v, \|v\|\tan\alpha)$ is a C^ω-diffeomorphism of $\big(V \setminus \{0\}\big) \times (-\pi/2\,, \pi/2)$ onto $\big(V \setminus \{0\}\big) \times \mathbb{R}$. Compute the Jacobian determinant of this map.



(The coordinate hyperplanes $\alpha = \alpha_0 =$ const. are cones (without vertex).)

8.8. (Power-coordinates) Let I be a finiet index set. For a matrix $\alpha=(\alpha_{ij})\in \mathrm{M}_I(\mathbb{R})$, let F_α denote the C^ω- map $(\mathbb{R}_+^\times)^I \to (\mathbb{R}_+^\times)^I$ with $F_\alpha\big((t_j)_{j\in I}\big) = \Big(\prod_{j\in I} t_j^{\alpha_{ij}}\Big)_{i\in I}$.

a). The Jacobian determinant of F_α is $\mathrm{J}(F_\alpha\,;t) = (\mathrm{Det}\,\alpha)\prod_{j\in I} t_j^{-1+\sum_{i\in I}\alpha_{ij}}$. Show that F_α is a diffeomorphism if and only if α is invertible.

b). The map $\alpha\mapsto F_\alpha$ is an (injektive) homomorphism of the group $\mathrm{GL}_I(\mathbb{R})$ in the group of C^ω-diffeomorphisms of $(\mathbb{R}_+^\times)^I$ into itself.

(**Hint:** One can clearly see the map F_α, in the coordinates $x_i := \ln t_i$, $i\in I$, on $(\mathbb{R}_+^\times)^I$).

† **Carl Gustav Jacob Jacobi (1804-1851)** was born on 10 Dec 1804 in Potsdam, Prussia (now Germany) and died on 18 Feb 1851 in Berlin, Germany. Carl Jacobi came from a Jewish family but he was given the French style name Jacques Simon at birth. His father, Simon Jacobi, was a banker and his family were prosperous. Carl was the second son of the family, the eldest being Moritz Jacobi who eventually became a famous physicist. There was a sister, Therese Jacobi, and a third brother, Eduard Jacobi, who was younger than Carl. Eduard did not pursue an academic career, but followed instead his father's profession as a banker.

Jacobi's early education was given by an uncle on his mother's side, and then, just before his twelfth birthday, Jacobi entered the Gymnasium in Potsdam. While still in his first year of schooling, he was put into the final year class and hence when he was still only 12 years old yet he had reached the necessary standard to enter university. The University of Berlin, however, did not accept students below the age of 16, so Jacobi had to remain in the same class at the Gymnasium in Potsdam until the spring of 1821.

Of course, Jacobi pressed on with his academic studies despite remaining in the same class at school. He received the highest awards for Latin, Greek and history but it was the study of mathematics which he took furthest. By the time Jacobi left school he had been undertaking research on his own attempting to solve quintic equations by radicals.

Jacobi entered the University of Berlin in 1821, he chose mathematics, but this did not mean that he could attend high level courses in mathematics for at this time the standard of university education in mathematics in Germany was rather poor. As he had done at the Gymnasium, Jacobi had to study on his own reading the works of Lagrange and other leading mathematicians. By the end of academic year 1823-24 Jacobi had passed the examinations necessary for him to be able to teach mathematics, Greek, and Latin in secondary schools. In 1825, he was offered a teaching post at the Joachimsthalsche Gymnasium, one of the leading schools in Berlin. He had submitted his doctoral dissertation to the University of Berlin even before he received the offer of the teaching post, and he was allowed to move quickly to work on his habilitation thesis.

Jacobi presented a paper concerning iterated functions to the Academy of Sciences in Berlin in 1825. However, the referees did not consider the results worth publishing and indeed the paper was not published by the Berlin Academy of Sciences. The paper was published eventually in 1961. Although this was not the best start for the young Jacobi, it did not hold him back for long and his publication record over the following years would be quite remarkable for both the number and quality of the works.

Around 1825 Jacobi changed from the Jewish faith to become a Christian which now made university teaching possible for him. By the academic year 1825-26 he was teaching at the University of Berlin. However prospects in Berlin were not good so, after taking advice from colleagues, Jacobi moved to the University of Königsberg. There he joined FRANZ NEUMANN, who had also received his doctorate from Berlin in 1825, and BESSEL who was the professor of astronomy at Königsberg.

Jacobi had already made major discoveries in number theory before arriving in Königsberg. He now wrote to GAUSS to tell him of the results on cubic residues which he had obtained, having been inspired by Gauss's results on quadratic and biquadratic residues. Gauss was impressed, so much so that he wrote to Bessel to obtain more information about the young Jacobi. But Jacobi also had remarkable new ideas about elliptic functions (as Abel did quite independently and at much the same time). On 5 August 1827 Jacobi wrote to LEGENDRE who was the leading expert on the topic.

Legendre immediately realised that Jacobi had made fundamental advances in his favourite topic. One would have to say that Legendre reacted extremely well to the realisation that his position as the leading expert on elliptic functions had changed overnight with the new theory being developed not only by Jacobi, but also by Abel. Jacobi's promotion to associate professor on 28 December 1827 was mainly due to the praise heaped on him by Legendre. In a letter, sent to Jacobi on 9 February 1828, Legendre wrote:

It gives me great satisfaction to see two young mathematicians such as you and [Abel] cultivate with such success a branch of analysis which for such a long time has been my favourite topic of study but which had not been received in my own country as well as it deserves. By your works you place yourselves in the ranks of the best analysts of our era.

In 1829 Jacobi met Legendre and other French mathematicians such as FOURIER and |sevensc Poisson when he made a visit to Paris in the summer vacation. On the journey to Paris he had visited GAUSS in Göttingen. Jacobi's fundamental work on the theory of elliptic functions, which had so impressed Legendre, was based on four theta functions. His paper Fundamenta nova theoria functionum ellipticarum published in 1829, together with its later supplements, made fundamental contributions to this theory of elliptic functions. However, despite Jacobi's brilliant contributions to elliptic functions he did not have the field to himself. As we have noted above, Abel was also making fundamental contributions and to some extent a competition had developed between the two. Legendre expressed this clearly in a letter he wrote to Jacobi early in 1829:

You proceed so rapidly, gentlemen, in all these wonderful speculations that it is nearly impossible to follow you - particularly for an old man ... I congratulate myself that I have lived long enough to witness these magnanimous contests between two young equally strong athletes, who turn their efforts to the profit of the science whose limits they push back further and further.

A few weeks after Legendre wrote this letter Abel died. On 11 September 1831 Jacobi married Marie Schwinck then, a few months later in May 1832, he was promoted to full professor after being subjected to a four hour disputation in Latin. Jacobi's reputation as an excellent teacher attracted many students. He introduced the seminar method to teach students the latest advances in mathematics. Jacobi had a major impact on these students and all others around him. Jacobi's forceful personality and sweeping enthusiasm that none of his gifted students could escape his spell: they were drawn into his sphere of thought, and soon represented a "school". C

W Borchardt, E Heine, L O Hesse, F J Richelot, J Rosenhain, and P L von Seidel belonged to this circle; they contributed much to the dissemination not only of Jacobi's mathematical creations but also the new research-oriented attitude in university instruction. The triad of Bessel, Jacobi, and Franz Neumann thus became the nucleus of a revival of mathematics at German universities.

In 1833 Jacobi's older brother Moritz joined him in Königsberg where he set himself up as an architect. During the two years Moritz spent there he became more interested in physics and left Königsberg in 1835 when he was appointed to the chair of civil engineering at Dorpat. In 1834 Jacobi received some work from Kummer who was at this time a teacher in a Gymnasium in Liegnitz, Jacobi immediately recognised Kummer's mathematical talents. Kummer had made advances beyond what Jacobi had achieved on third-order differential equations and Jacobi wrote to his brother Moritz in 1836 describing how Kummer had managed to solve problems which had defeated him.

In 1834 Jacobi proved that if a single-valued function of one variable is doubly periodic then the ratio of the periods is imaginary. This result prompted much further work in this area, in particular by Liouville and Cauchy. Jacobi carried out important research in partial differential equations of the first order and applied them to the differential equations of dynamics. He also worked on determinants and studied the functional determinant now called the Jacobian. Jacobi was not the first to study the functional determinant which now bears his name, it appears first in a 1815 paper of Cauchy. However Jacobi wrote a long memoir *De determinantibus functionalibus* in 1841 devoted to this determinant. He proved, among many other things, that if a set of n functions in n variables are functionally related then the Jacobian is identically zero, while if the functions are independent the Jacobian cannot be identically zero.

One of the prettiest results in the global theory of curves is a theorem of Jacobi (1842): *The spherical image of the normal directions along a closed differentiable curve in space divides the unit sphere into regions of equal area.* The statement of this theorem is an afterthought to a paper in which Jacobi responds to the published correction by Thomas Clausen (1842) of an earlier paper by Jacobi (1836).

In July 1842 Jacobi and Bessel attended the meeting of the British Association for the Advancement of Science in Manchester as representatives of Prussia. Jacobi's wife accompanied the two mathematicians. They returned to Königsberg via Paris where Jacobi lectured at the Académie des Sciences. In the following year Jacobi became unwell and diabetes was diagnosed. He was advised by his doctor to spend time in Italy where the climate would help him recover. However, Jacobi was not a wealthy man and Dirichlet, after visiting Jacobi and discovering his plight, wrote to Alexander von Humboldt asking him to help obtain some financial assistance for Jacobi from Friedrich Wilhelm IV.

We should make a small digression to say why Jacobi was not a wealthy man despite having inherited a small fortune from his wealthy father. A severe business depression throughout Prussia (in fact it was a Europe wide depression), had led to a bankruptcy in which Jacobi had lost all his money. Let us now return to Dirichlet and Alexander von Humboldt's attempts to help obtain support for Jacobi's trip to Italy.

Jacobi had frequently corresponded with Alexander von Humboldt. The correspondence began in 1828 but only after 1839 did they correspond regularly and the 44 surviving letters between the two men make fascinating reading. Dirichlet's request to Friedrich Wilhelm IV, supported strongly by Alexander von Humboldt, was successful and Jacobi received a grant to allow him to spend time in Italy. He set off for Italy with Borchardt and Dirichlet and, after stopping in several towns and attending a mathematical meeting in Lucca, they arrived in Rome on 16 November 1843. Schläfli and Steiner were also with them, Schläfli being their interpreter.

The climate in Italy did indeed help Jacobi to recover and he began to publish again, his health having prevented him working for some time before this. In fact Jacobi's interests in mathematics were very wide and while in Rome he took the opportunity to satisfy his interest in the history of mathematics working on manuscripts of Diophantus's Arithmetica which were kept in the Vatican. Although his health had improved it was felt that the climate of Königsberg was too extreme for him to return there, so a dispensation was obtained from Friedrich Wilhelm IV to allow him to transfer to Berlin. He was given a supplement to his salary to help offset the higher costs of living in Berlin, and also to help him with his medical expenses.

He was in Berlin by June 1844 and although his health prevented him from giving frequent lecture courses, he did lecture at the University of Berlin. Jacobi in his lectures on analytical mechanics (Berlin, 1847 - 1848) ... gave a detailed and critical discussion of Lagrange's mechanics. Lagrange's view that mechanics could be pursued as an axiomatic-deductive science forms the centre of Jacobi's criticism and is rejected on mathematical and philosophical grounds. ... Jacobi's criticism is motivated by a changed evaluation of the role of mathematics in the empirical sciences. Jacobi only came to hold these views on analytical mechanics only later in his life, for earlier he had ignored the physical interpretation of mechanics in favour of a purely axiomatic and mathematical approach.

By 1848 conditions were bad in the German Confederation. Unemployment and crop failures had led to discontent and disturbances. The news that Louis-Philippe had been overthrown by an uprising in Paris in February 1848 led to revolutions in many states and fighting in Berlin. Republican and socialist feelings meant that the monarchy was in trouble. Jacobi made a political speech in the Constitutional Club in Berlin which managed to upset both the monarchists and the republicans. As a consequence Jacobi's request to be allowed to join the staff of the University of Berlin was refused by the Prussian government.

By the summer of 1849 the revolution was completely defeated. The Prussian government, still feeling aggrieved at Jacobi, took away the supplement to his salary which allowed him to live in Berlin. He had to move, and chose the small town of Gotha. He lived there with his family and a few months later accepted a chair at the University of Vienna. The Prussian government suddenly realised what they would lose if they forced Jacobi to leave Prussia, so they made concessions which meant that Jacobi could lecture at the University of Berlin while his family remained in Gotha. It was not a good deal for Jacobi and the fact that he accepted it means that he was strongly attached to his own country.

Jacobi planned to spend the university vacations with his family and he spent the summer of 1850 with them in Gotha. In January 1851 he contracted influenza, then he contracted smallpox before he had regained his strength. He died a few days after contracting smallpox.