MA-302 Advanced Calculus

9. Constant rank theorem

Hermann Amandus Schwarz†
(1843-1931)

In the following V and W denote finite dimensional $\mathbb{K}$-vector spaces, $G \subseteq V$ an open subset and $F : G \to W$ a C^k-map, $k \in \mathbb{N}^* \cup \{\infty, \omega\}$.

9.1. If F is submersion, then show that F is an open map, i.e. for every open subset U in G the image $F(U)$ is open in W.

9.2. a). The subset U of the points $x \in G$, at which F is subimmersive, is open and dense in G. (**Hint:** It is enough to prove that $U \neq \emptyset$. Consider the subset $\{x \in G \mid \text{Rank}\,(\mathrm{D}F)_x = m\}$, where m is the maximum of the ranks of $(\mathrm{D}F)_x$, $x \in G$.)

b). Suppose that F is open. Then F is subimmersive at $x \in G$ if and only if F is submersive at x. Further, show that the subset of the points $x \in G$, at which F is submersive, is open and dense in G.

c). Suppose that F is injective. Then show that F is subimmersive at $x \in G$ if and only if F is immersive at x. Further, show that the subset of the points $x \in G$, at which F is immersive is open and dense in G.

9.3. An injective immersion $F : G \to W$ is an embedding if and only if for every point $a \in G$ and every opne neighbourhood U of a in G, there exists an open neighbourhood U' of $F(a)$ in W such that $F(G \setminus U) \cap U' = \emptyset$.

9.4. Let $F : G \to W$ be an injective immersion. Show that F induces an embedding on every relatively comptact open subset $U \subseteq G$, which is contained in a relatively compact subset of G (then $\overline{U} \subseteq G$ is compact).

9.5. Let $F_1, \ldots, F_m$ be C^k-functions on G, $k \in \mathbb{N}^* \cup \{\infty, \omega\}$. Suppose that in a neighbourhood of a point $a \in G$ the total differential of $F = (F_1, \ldots, F_m) : G \to \mathbb{K}^m$ has the constant rank p. Show that there exsits a regular C^k-function $H : U' \to \mathbb{K}$ in a neighbourhood U' of $F(a)$ such that $H\big(F_1(x), \ldots, F_m(x)\big) = 0$ for all x in a neighbourhood U of a, if and only if $p < m$. (In this case one says that the functions $F_1, \ldots, F_m$ are C^k-*dependent* at the point a.)

9.6. Let M be a C^k-submanifold of V and let N be a C^k-submanifold of W. Show that $M \times N$ is a C^k-submanifold of $V \times W$.

9.7. Let $L_1, \ldots, L_r$ be linear forms on the real vector space V and let $f : G \to \mathbb{R}$ be a C^1-function. If f has a constrained local extremum at $a \in G$ on the linear subspace $L_1 = b_1, \ldots, L_r = b_r$, then show that $(df)_a$ is a linear combination of $L_1, \ldots, L_r$, i.e. $(df)_a = \sum_{i=1}^{r} \lambda_i L_i$. (The λ_i are uniquely determined if the L_i are linearly independent, which can be assumed without losss of generality.)

† **Hermann Amandus Schwarz (1843-1931)** was born on 25 Jan 1843 in Hermsdorf, Silesia (now Poland) and died on 30 Nov 1921 in Berlin, Germany.

Hermann Schwarz's father was an architect. He studied at the Gymnasium in Dortmund where his favourite subject was chemistry. When he left school he intended to take a degree in chemistry and he entered the Gewerbeinstitut, later called the Technical University of Berlin, with this aim. Schwarz began his study of chemistry at Berlin but it was not long before KUMMER and WEIERSTRASS had influenced him to change to mathematics. The first of his teachers to influence the direction that his research would eventually take was KARL POHLKE. Through him Schwarz became interested in geometry. Schwarz attended Weierstrass's lectures on The integral calculus in 1861 and the notes that Schwarz took at these lectures still exist. His interest in geometry was soon combined with Weierstrass's ideas of analysis. He continued to study in Berlin, being supervised by Weierstrass, until 1864 when he was awarded his doctorate. His doctoral dissertation was examined by Kummer.

While in Berlin, Schwarz worked on minimal surfaces (surfaces of least area), a characteristic problem of the calculus of variations. PLATEAU published a famous memoir on the topic in 1866 and in the same year Weierstrass established a bridge between the theory of minimal surfaces and the theory of analytic functions. Schwarz had made an important contribution in 1865 when he discovered what is now known as the Schwarz minimal surface. This minimal surface has a boundary consisting of four edges of a regular tetrahedron.

Schwarz continued studying in Berlin for his teacher's training qualification which he completed by 1867. In that year he was appointed as a Privatdozent to the University of Halle. In 1869 he was appointed as professor of mathematics at the Eidgenössische Technische Hochschule in Zurich then, in 1875, he accepted appointment to the chair of mathematics at Göttingen University.

Perhaps surprisingly after Schwarz succeeded Weierstrass accepting a professorship in Berlin in 1892, the balance in favour of the most eminent university in Germany for mathematics, which had undoubtedly been Berlin, began to shift towards Göttingen. There were several reasons for this. Firstly Schwarz failed to keep up his output of mathematical research after his move. Bieberbach in [2] put it rather well when he wrote that Schwarz retired to Berlin in 1892. That this was the case should not have come as a complete surprise to those making the appointment for Schwarz had published his Complete Works in 1890, two years earlier. ... *teaching duties and concern for [Schwarz's] many students took so much of his time that he published very little more. A contributing element may have been his propensity for handling both the important and the trivial with the same thoroughness, a trait also evident in his mathematical papers.*

We should not give the impression that the only reason for Berlin moving down from being the leading German university for mathematics to become its second university was due to Schwarz. The other effect was KLEIN whose dynamic leadership in Göttingen made it prosper at the expense of Berlin where Frobenius and Schwarz could not provide the same inspired approach. Perhaps the final sign that Göttingen had overtaken Berlin came in 1902 when FROBENIUS and Schwarz chose HILBERT to succeed to the Berlin chair which had become vacant on the death of FUCHS. Hilbert turned down the offer, preferring to remain at Göttingen. The Berlin chair was then filled by SCHOTTKY but, like Schwarz before him, he had moved to Berlin after his best days for mathematical research were behind him.

Schwarz continued teaching at Berlin until 1918. We shall describe some of his very fine mathematical achievements in a moment, but first we note that he had several interests outside mathematics, although his marriage was a mathematical one since he married Kummer's daughter. Outside mathematics he was the captain of the local Voluntary Fire Brigade and, more surprisingly, he assisted the stationmaster at the local railway station by closing the doors of the trains.

One important area which Schwarz worked on was that of conformal mappings. In 1870 he produced work related to the Riemann mapping theorem. Although RIEMANN had given a proof of the theorem that any simply connected region of the plane can be mapped conformally onto a disc, his proof involved using the Dirichlet problem. Weierstrass had shown that Dirichlet's solution to this was not rigorous. Schwarz's gave a method to conformally map polygonal regions to the circle. Then, by approximating an arbitrary simply connected region by polygons he was able to give a rigorous proof of the Riemann mapping theorem. Schwarz also gave the alternating method for solving the Dirichlet problem which soon became a standard technique.

His most important work is a Festschrift for Weierstrass's 70th birthday. Schwarz answered the question of whether a given minimal surface really yields a minimal area. An idea from this work, in which he constructed a function using successive approximations, led Emile Picard to his existence proof for solutions of differential equations. It also contains the inequality for integrals now known as the "Schwarz inequality".

The fact that Schwarz should have come up with a special case of the general result now known as the Cauchy-Schwarz inequality (or the Cauchy-Bunyakovsky-Schwarz inequality) is not surprising for much of his work is characterised by looking at rather specific and narrow problems but solving them using methods of great generality which have since found widespread applications. That he found such general methods says much for his great intuition which was perhaps based on a deep feeling for geometry.

For example the Cauchy-Schwarz inequality appears in arithmetic, geometric and function-theoretic formulations in works of mathematicians such as BUNYAKOVSKY, CAUCHY, GRASSMANN, VON NEUMANN and WEYL.

In answering the problem of when Gauss's hypergeometric series was an algebraic function Schwarz, as he had done so many times, developed a method which would lead to much more general results. It was in this work that he defined a conformal mapping of a triangle with arcs of circles as sides onto the unit disc which is now known as the 'Schwarz function'. This function is an early example of an automorphic function and in this work Schwarz was looking at ideas which led KLEIN and POINCARÉ to develop the theory of automorphic functions.

Schwarz was deeply influenced by Weierstrass. From their correspondence one finds that Schwarz addressed his teacher often with an accuracy going down to the last detail, sometimes almost timidly. Schwarz's demeanour has been described as naive, dramatic, coarse. In spite of giving the impression of self-confidence, he was, in fact, rather insecure and besides, not efficient in business matters.