MA-302 Advanced Calculus

11. Differential forms

Johann Friedrich Pfaff†
(1765-1825)

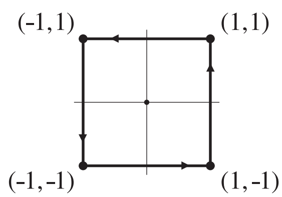
The exercises 11.2 and 11.7 are only to get practice, their solutions need not be submitted.

11.1. Let $P_{2+n} : \mathbb{R}^{2+n} \to \mathbb{R}^{2+n}$ be the polar coordinate map [1]) in the dimension $2+n$. For a 1–form $\omega := \sum_{i=1}^{2+n} F_i\, dx_i$ on $\mathbb{R}^{2+n}$, where x_i are the standard coordinate functions, write explicitly the pull-back 1–form $P_{2+n}^{*}\omega$ on $\mathbb{R}^{2+n}$.

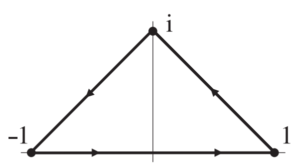
11.2. Compute the path integral $\int_\gamma \omega$ for the following ω and γ :

a). $\omega = y^2\, dx + dy$ on $\mathbb{R}^2$ and for $\alpha \geq 1$, $\gamma_\alpha(t) : [0,1] \to \mathbb{R}^2$ defined by $\gamma_\alpha(t) = (t, t^\alpha)$.

b). $\omega := \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ and γ is the boundary of the following square in $\mathbb{R}^2$.



c). $\omega := |z|\, dz$ on $\mathbb{C}$ and γ is the boundary of the following triangle in $\mathbb{C}$.



11.3. Let $\gamma_r : [0, 2\pi] \to \mathbb{R}^2$ be the circle $\gamma_r(t) := r(\cos t, \sin t)$ of radius $r > 0$ with center 0 . For the winding form $\omega := \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ and every continuous function f in a neighbourhood of $0 \in \mathbb{R}^2$, show that $f(0) = \frac{1}{2\pi} \lim_{r\to 0+} \int_{\gamma_r} f\omega$.

11.4. Let $H : G' \to G$ be a C^1-map from the open subset $G' \subseteq V'$ with values in the open subset G, ω be a continuous 1-form on G and $H^*\omega$ be the corresponding (continuous) 1-form on G'. If ω is exact resp. closed, then so is $H^*\omega$.

11.5. Show that the C^∞–map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (-y, x)$ does not have primitive on $\mathbb{R}^2$.

11.6. Show that

[1]) $P_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $P_2(r,\varphi) := (r\cos\varphi, r\sin\varphi)$ and $P_{2+n} : \mathbb{R}^{2+n} \to \mathbb{R}^{2+n}$ is defined recursively by $P_{2+n}(r, \varphi_0, \ldots, \varphi_n) := (P_{2+n-1}(r, \varphi_0, \ldots, \varphi_{n-1}) \cdot \cos\varphi_n\, ,\ r \cdot \sin\varphi_n)$.

a). the winding form $\omega := \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$ on $\mathbb{R}^2 \setminus \{0\}$ is closed and not exact.

b). the form $\omega = \dfrac{(x^2-y^2)}{(x^2+y^2)^2}\,dx + \dfrac{2xy}{(x^2+y^2)^2}\,dy$ on $\mathbb{R}^2 \setminus \{0\}$ is exact.

11.7. Show that the following diferential forms are exact and find their primitives.

a). $(y^2e^{xy} + 3x^2y)\,dx + \big(x^3 + (1+xy)\,e^{xy}\big)\,dy$ on $\mathbb{R}^2$.

b). $(x-y)\,dx + \big(\frac{1}{y^2} - x\big)\,dy$ on $\mathbb{R} \times \mathbb{R}_+^\times$.

c). $\big(\frac{1}{2}\,y^2e^x + 2ye^{2x}\big)\,dx + \big(ye^x + e^{2x}\big)\,dy$ on $\mathbb{R}^2$.

d). $(x+z)\,dx + (-y-z)\,dy + (x-y)\,dz$ on $\mathbb{R}^3$.

e). $(2x+y)\,dx + (x+z^2)\,dy + 2yz\,dz$ on $\mathbb{R}^3$.

f). $(\sin y + y\cos x + yz\cos xy)\,dx + (x\cos y + \sin x + xz\cos xy)\,dy + \sin xy\,dz$ on $\mathbb{R}^3$.

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† **Johann Friedrich Pfaff (1765-1825)** was born on 22 Dec 1765 in Stuttgart, Württemberg (now Germany) and died on 21 April 1825 in Halle, Saxony (now Germany). Johann Friedrich Pfaff's father, Burkhard Pfaff, was chief financial councillor of Württemberg while his mother was the daughter of a member of the exchequer of Württemberg. It was a family with a tradition of working as civil servants for the government of Württemberg.

Johann Friedrich was the second of his parents seven sons and, although perhaps the one to attain the greatest fame, he was certainly not the only one to excel in science. The youngest of the family, JOHANN WILHELM PFAFF who was born in 1774, also became a mathematician and held chairs in Würtzburg and Erlangen. The second youngest, CHRISTOPH HEINRICH PFAFF was born in 1773 and, with interests in chemistry, medicine and pharmacy, he worked with Volta on electricity in animals.

There was a school in Stuttgart, the Hohe Karlsschule, which was run to train sons of government officials of Württemberg and Johann Friedrich attended this school from the age of nine. It was a rather uninspiring school, strong on discipline but less good academically. Pfaff did not learn much in the way of mathematics there despite attending the school until he was nearly twenty. When he left in the autumn of 1785 he had completed his studies in law, a fitting subject for a civil servant.

Despite a lack of training in mathematics at his school, Pfaff had studied mathematics on his own and began to study the works of Euler. He was encouraged to move toward scientific topics by the Duke of Württemberg, and he spent two year studying at the University of Göttingen where he was taught mathematics by KAESTNER and he also studied physics. From Göttingen, Pfaff moved to Berlin in the summer of 1787. There he studied astronomy under J E BODE, and Pfaff wrote his first paper which was on a problem in astronomy.

In the spring of 1788 Pfaff set off on a journey to Vienna but he visited many universities on the way, in particular Halle, Jena, Helmstedt, Gotha, Dresden, and Prague. Klügel was professor of mathematics at Helmstedt and he accepted a chair at Halle leaving the position at Helmstedt vacant. Pfaff's physics professor at Göttingen recommended him for the chair, and Pfaff submitted a dissertation on the occasion of his election as professor of mathematics at the University of Helmstedt. It was a tradition that new professors at the university there submitted an inaugural dissertation.

Pfaff's inaugural dissertation was titled Programma inaugurale in quo peculiarem differentialia investigandi rationem ex theoria functionum deducit. It investigates the use of some functional equations in order to calculate the differentials of logarithmic and trigonometrical functions as well as the binomial expansion and Taylor formula. This is studied in detail in [4].

From his appointment in 1788 until 1810 Pfaff held the chair at Helmstedt. His appointment was approved by the Duke of Württemberg but it was not the best of positions, since it was poorly paid. He did good work building the strength of mathematics there and he put much effort into teaching and was successful in increasing the number of students of mathematics. One student who studied at Helmstedt was GAUSS. After studying at Göttingen, Gauss came to Helmstedt in 1798. He attended Pfaff's lectures and even lived in his house. Pfaff recommended Gauss's doctoral dissertation and, when necessary, greatly assisted him; Gauss always retained a friendly memory of Pfaff both as a teacher and as a man.

By the time Gauss studied with Pfaff at Helmstedt, the university was under threat of closure. Pfaff fought hard to prevent this and for a few years he was successful. Pfaff married in 1803 to CAROLINE BRAND but sadly their first child died as an infant. By 1810 Pfaff's attempts to preserve the University of Helmstedt finally failed with the closure of the university. The staff were given a number of different choices as to which university they might move to, and Pfaff chose to move to Halle. He was appointed to the chair of mathematics at Halle in 1810 and in 1812, on the death of KLÜGEL, he took on the directorship of the University Observatory too.

Pfaff did important work in analysis working on partial differential equations, special functions and the theory of series. He developed Taylor's Theorem using the form with remainder as given by Lagrange. In 1810 he contributed to the solution of a problem due to Gauss concerning the ellipse of greatest area which could be drawn inside a given quadrilateral. His most important work on Pfaffian forms was published in 1815 when Pfaff was nearly fifty years old but its importance was not recognised until 1827 when JACOBI published a paper on Pfaff's method. This failure to recognise the importance of the work is strange, particularly given the very positive review which Gauss wrote of the work shortly after it was published. In the 1815 paper, which Pfaff submitted to the Berlin Academy on 11 May, he presented a transformation of a first-order partial differential equation into a differential system. This theory of equations in total differentials is undoubtedly Pfaff's most significant contribution. ... constituted the starting point of a basic theory of integration of partial differential equations which, through the work of Jacobi, Lie, and others, has developed into a modern Cartan calculus of extreme differential forms.

Among his other important works are Disquisitiones analytica maxime ad calculum integralem et doctrinam serierum pertinentes (1797), an introductory work written in the style of Euler, and Observationes ad Euleri institutiones calculi integralis.