MA-231 Topology

2. Finite Sets

August 13, 2004 ; Submit solutions **before 10:00 AM ; August 23, 2004.**

2.1. Let X be a finite set with n elements. For $i \in \mathbb{N}$, let $\mathfrak{P}_i(X)$ be the set of all subsets Y of X with $|Y| = i$. Show that: If $i \in \mathbb{N}$ with $0 \le i < n/2$ (resp. with $n/2 < i \le n$), then there exists an injective map $f_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i+1}(X)$ such that $Y \subseteq f_i(Y)$ for all $Y \in \mathfrak{P}_i(X)$ (resp. an injective map $g_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i-1}(X)$ such that $g_i(Y) \subseteq Y$ for all $Y \in \mathfrak{P}_i(X)$). (**Hint:** Let $0 \le i < n/2$. A pair $(Y, Y') \in \mathfrak{P}_i(X) \times \mathfrak{P}_{i+1}(X)$ is called *amicable* if $Y \subseteq Y'$. Let $\mathfrak{R}$ be a subset of $\mathfrak{P}_i(X)$ with $|\mathfrak{R}| =: r$. Further, let $\mathfrak{R}'$ be the set of all those $Y' \in \mathfrak{P}_{i+1}(X)$ which are amicable to at least one $Y \in \mathfrak{R}$. Put $s := |\mathfrak{R}'|$. Then $r(n-i) \le s(i+1)$ and hence $r \le s$. Now use the Marriage-theorem[1]))

2.2. Let $X_1, \ldots, X_n$ be finite sets. For $J \subseteq \{1, \ldots, n\}$, let $X_J := \bigcap_{i \in J} X_i$ with $X_\emptyset := \bigcup_{i=1}^n X_i$. Generalize the formula $|Y \cup Z| = |Y| + |Z| - |Y \cap Z|$ for finite sets Y, Z, prove the well-known Sylvester's (Sieve–) formula[2]):

$$\sum_{J \in \mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|} |X_J| = 0\,, \quad \text{i.e.} \quad |X| = \sum_{\emptyset \ne J \in \mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|-1} |X_J|\,.$$

(**Hint:** By induction on n. — Variant: For $k = 1, \ldots, n$, let Y_k be the set of elements $x \in X_\emptyset$ which belong to exactly k of the sets $X_1, \ldots, X_n$. Then Y_k, $1 \le k \le n$ are pairwise disjoint. Using Exercise T2.2 b) show that

$$\sum_{\substack{J \in \mathfrak{P}(\{1,\ldots,n\}) \\ |J| \text{ even}}} |X_J| = \sum_{k=1}^{n} 2^{k-1} |Y_k| = \sum_{\substack{J \in \mathfrak{P}(\{1,\ldots,n\}) \\ |J| \text{ odd}}} |X_J|\,.)$$

2.3. a). Let X be a finite set with m elements. Let p_m denote the number of permutations of X which donot have fixed points and let $s_m = m!$ be the number of all all permutations of X. Show that:

$$\frac{p_m}{s_m} = \frac{1}{0!} - \frac{1}{1!} + \cdots + (-1)^m \cdot \frac{1}{m!}\,.$$

(**Hint:** Let $X = \{x_1, \ldots, x_m\}$. Set $X_i := \{\sigma \in \mathfrak{S}(X) : \sigma(x_i) = x_i\}$ and compute $s_m - p_m = |\bigcup_{i=1}^m X_i|$ using the Sieve formula in Exercise 2.2. — **Remark :** Note that $\lim_{m\to\infty}(p_m/s_m) = e^{-1}$, where $e = 2,718\ldots$ is the base of the natural logarithm.) — The number of permutations of X with exactly r fixed points is $\binom{m}{r} p_{m-r}$, $0 \le r \le m$. (Proof!)

b). Let X be a finite set with m elements and let Y be a finite set with n elements. The number of surjective maps from X in Y is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n \binom{n}{n}(n-n)^m\,.$$

(**Hint:** Let $Y = \{y_1, \ldots, y_n\}$. Set $P_i := \{f \in Y^X : y_i \notin \operatorname{im} f\}$ and compute the number $|\bigcup_{i=1}^n P_i|$ of non-surjective maps using the Sieve formula in Exercise 2.2.)

2.4. Let I be a finite index set with n elements and let $\sigma_i \in \mathbb{N}$ for $i \in I$, $\pi := \prod_{i \in I} \sigma_i$, $\sigma := \sum_{i \in I} \sigma_i$ and $\sigma_H := \sum_{i \in H} \sigma_i$ for $H \subseteq I$. Then

$$\sum_{H \subseteq I} (-1)^{|H|} \binom{\sigma_H}{n} = (-1)^n \pi \quad \text{and} \quad \sum_{H \subseteq I} (-1)^{|H|} \binom{\sigma_H}{n+1} = \frac{(-1)^n}{2}(\sigma - n)\pi\,,$$

(**Hint:** Let $X = \bigcup_{i \in I} X_i$, where X_i are pairwise disjoint subsets with $|X_i| = \sigma_i$. For a proof of the first formula consider the set $\mathfrak{P}_n(X)$ and its subsets $Y_i := \{A \in \mathfrak{P}_n(X) \mid A \cap X_i = \emptyset\}$ and use the Sieve formula in Exercise 2.2 to find $|\bigcup_{i \in I} Y_i|$.)

On the other side one can see (simple) test-exercises ; their solutions need not be submitted.

[1]) **Marriage-theorem :** *Let Y_x, $x \in X$, be a finite family of sets. For every subset N of X assume that the set $Y_N := \cup_{x \in N} Y_x$ has atleast $|N|$ elements. Then there exists an injective choice function $f : X \to Y_X$ with $f(x) \in Y_x$ for every $x \in X$.*

[2]) This is also called the Inclusion-Exclusion principle

Test-Exercises

T2.1. (Indicator functions) Let I be a set. For a subset $J \in \mathfrak{P}(I)$, let $e_J : X \to \{0,1\}$ be the indicator function of J (with respect to I), i.e. $e_J(i) = \begin{cases} 1, & \text{if } i \in J, \\ 0, & \text{if } i \in I \setminus J. \end{cases}$ Note that $e_I = 1$ and $e_\emptyset = 0$. Show that

a). The map $J \mapsto e_J$ is a bijective map from the poer set $\mathfrak{P}(I)$ onto the set $\{0,1\}^I$ of all maps $I \to \{0,1\}$.

b). For subsets $J, K \subseteq I$, prove that: $e_{J\cap K} = e_J e_K$, $e_{J\cup K} = e_J + e_K - e_J e_K$, $e_{J\setminus K} = e_J(1-e_K)$. In particular, $e_{I\setminus J} = 1 - e_J$ and $e_{J \triangle K} = e_J + e_K - 2e_J e_K$.

c). For $J, K \in \mathfrak{P}(I)$, let $J + K := J \Delta K := (J \cup K) \setminus (J \cap K)$ denote the symmetric difference of J and K. Then show that

1) $J + K = K + J$ and $J + \emptyset = J$, $J + J = \emptyset$.

2) $(J+K)+L = J+(K+L)$ for all $J, K, L \in \mathfrak{P}(I)$.

3) For every $J, L \in \mathfrak{P}(I)$, there exists a unique K such that $J + K = L$.

4) $(J+K)\cap L = (J\cap L) + (K \cap L)$ for all $J, K, L \in \mathfrak{P}(I)$.

— **Remark:** For verification of these properties use indicator functions and their rules given in b). These properties of the symmetric difference $\triangle$ show that the power set $\mathfrak{P}(I)$ with the symmetric difference Δ as addition and the intersection $\cap$ as multiplication is a commutative ring with $\emptyset$ as the zero element 0 and I as the unit element 1. This ring is called the set-ring of I. If $|I| = 1$, then this ring is a field with two elements; in the other case the set-ring of I is not a field.

T2.2. Let X be a finite set with n elements.

a). The number of subsets of X is 2^n (Induction).

b). If $n \in \mathbb{N}^*$, then the number of subsets of X with an even number of elements is equal to the number of subsets of X with an odd number of elements. Moreover, this number is equal to 2^{n-1}. (**Hint:** Let $a \in X$. The map defined by $A \mapsto A \cup \{a\}$, if $a \notin A$, resp. $A \setminus \{a\}$, if $a \in A$, is a bijective map from the set of subsets with an even number of elements onto the set of subsets with an odd number of elements.)

T2.3. a). From 1a) deduce that: For $n \in \mathbb{N}$, $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

b). From 1b) deduce that: For $n \in \mathbb{N}^*$, $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0$.

c). Let X be a finite set with n elements. The number of pairs (X_1, X_2) in $\mathfrak{P}(X) \times \mathfrak{P}(X)$ with $X_1 \cap X_2 = \emptyset$ is 3^n (Induction). General: The number of r–tuples $(X_1, \ldots, X_r)$ of pairwise disjoint subsets $X_1, \ldots, X_r \subseteq X$ is equal $(r+1)^n$, $r \in \mathbb{N}$.

d). For $m, n, k \in \mathbb{N}$, $\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0}$. In particular, $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ for $n \in \mathbb{N}$. (**Hint:** Let X, Y be disjoint sets with $|X| = m$, $|Y| = n$. The assignment $A \mapsto (A \cap X,\ A \cap Y)$ defines a bijective map $\mathfrak{P}(X \cup Y) \to \mathfrak{P}(X) \times \mathfrak{P}(Y)$.)

T2.4. Let m be a natural number (resp. a positive natural number) and let n be another natural number. Let $\mathrm{a}(m,n)$ (resp. $\mathrm{b}(m,n)$) denote the number of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \le n$ (resp. $x_1 + \cdots + x_m = n$). Show that

$$\mathrm{a}(m,n) = \binom{n+m}{m}, \quad \mathrm{b}(m,n) = \binom{n+m-1}{m-1}.$$

(**Hint:** Note that $\mathrm{a}(m-1,n) = \mathrm{b}(m,n)$ and $\mathrm{a}(m,n) = \mathrm{a}(m,n-1) + \mathrm{a}(m-1,n)$ if $m \ge 1$ and use induction on $n+m$. —Variant: The map $(x_1, \ldots, x_m) \mapsto \{x_1+1,\ x_1+x_2+2, \ldots, x_1+\cdots+x_m+m\}$ maps the set of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \le n$ bijectively onto the set of m–element subsets of $\{1, 2, \ldots, n+m\}$.)

T2.5. Let $\mathfrak{X} = (X_1, \ldots, X_r)$ and let $\mathfrak{Y} = (Y_1, \ldots, Y_r)$ be partitions of the set X into r pairwise disjoint subsets each of them with $n \ge 1$ elements (i.e. $\bigcup_{i=1}^r X_i = X$ and $X_i \cap X_j = \emptyset$ for $i \ne j$ and analogously for $\mathfrak{Y}$). Show that: $\mathfrak{X}$ and $\mathfrak{Y}$ has a common representative system, i.e. there exist r distinct elements $x_1, \ldots, x_r$ in X such that each x_i belongs to exactly one of the subset $X_1, \ldots, X_r$ and exactly one of the subset $Y_1, \ldots, Y_r$. (**Hint:** Using the Marriage-theorem find a permutation $\sigma \in \mathfrak{S}_r$ such that $X_i \cap Y_{\sigma(i)} \ne \emptyset$ for every $1 \le i \le r$.)