MA-231 Topology

8. Quotient Spaces[1])

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Waclaw Sierpinski†
(1882-1969)

8.1. (Strong Topology) Let $X_i, i \in I$ be topological spaces, Y be a set and let $f_i : X_i \to Y$.

a). Show that $\mathcal{T} := \{U \in \mathfrak{P}(Y) \mid f_i^{-1}(U)$ is open in X_i for each $i \in I\}$ is a topology on Y; This topology on Y is called the strong topology induced by the maps $f_i, i \in I$ and that this topology on Y is the largest making each f_i continuous.

b). Suppose that Y has the strong topology induced by the maps $f_i, i \in I$. Let $g : Y \to Z$ be a map into a topological space Z. Then g is continuous if and only if $g \circ f_i$ is continuous for each $i \in I$.

c). The family $f_i, i \in I$ of maps is said to cover points of Y if each $y \in Y$ is in the image of some f_i. For families which cover points, the strong topology is just a quotient topology, more precisely:

On the disjoint union [2]) $\sum_{i \in I} X_i$, define a relation $\sim$ by: $x \sim y$ if $f_i(x) = f_j(y)$, where $i, j \in I$ are indices such that $x \in X_i$ and $y \in X_j$. Then $\sim$ is an equivalence relation on $\sum_{i \in I} X_i$. Further, if the family $f_i, i \in I$ covers points of Y, then Y has the strong topology if and only if Y is homeomorphic to the quotient space $\sum_{i \in I} X_i/\sim$. (**Hint:** The map $h : Y \to X/\sim$ defined by $y \mapsto [x]$, where $y = f_i(x)$ for some $i \in I$ and $x \in \sum_{i \in I} X_i$, is a homeomorphism.)

8.2. Let $\{\mathcal{T}_i \mid i \in I\}$ be a family of topologies on a fixed set X and denote by X_i the topological space consisting of the set X with the topology $\mathcal{T}_i$. For each $i \in I$, let $\iota_i : X_i \to X$ be the identity map.

a). The strong topology on X induced by the maps $\iota_i, i \in I$ is the intersection (infimum) $\mathcal{T}$, of the topologies $\mathcal{T}_i$.

b). On the disjoint union $\sum_{i \in I} X_i$, define a relation $\sim$ by: $x \sim y$ if $\iota_i(x) = \iota_j(y)$, where $i, j \in I$ are indices such that $x \in X_i$ and $y \in X_j$. Then $\sim$ is an equivalence relation on $\sum_{i \in I} X_i$ and that the quotient space is homeomorphic to the topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is the topology given in the part a).

8.3. (Disjoint union and products) Let $X_i, i \in I$ be topological spaces. If X_i is homeomorphic to a (given) topological space X for each $i \in I$, then the disjoint union $\sum_{i \in I} X_i$ is homeomorphic to the product space $X \times I$ where the indexed set I is given the discrete topology.

8.4. Let X, Y be two sets and let $f : X \to Y$ be a map. A subset $A \subseteq X$ is called f-saturated if it is the inverse image of some subset $B \subseteq Y$, i.e. $A = f^{-1}(B)$. Note that a subset A of X is f-saturated if and only if $A = f^{-1}(f(A))$. Further, note that for any subset A of X, the subset $f^{-1}(f(A))$ is

[1]) The Quotient topology was first studied by Moore, R. L. in 1925 and Alexandroff, P. in 1926

[2]) **Disjoint Union** (or **Free union**) Let $X_i, i \in I$ be topological spaces and let $X_i^* := X_i \times \{i\}$ with the topology being defined on X_i^* in the obvious way, to make it homeomorphic to X_i. Then $X_i^* \cap X_j^* = \emptyset$ if $i \neq j$. Define a topology on $\cup_{i \in I} X_i^*$ as follows: $U \subseteq \cup_{i \in I} X_i^*$ is open if and only if $U \cap X_i^*$ is open for each $i \in I$. The resulting topological space is called the disjoint union or free union of the family of topological spaces X_i $i \in I$ and is denoted by $\sum_{i \in I} X_i$. If only two topological spaces X and Y are involved, then we write $X + Y$ for the disjoint union of X and Y.

f-saturated subset containing A. For any subset A of X, the f-saturated subset $f^{-1}(f(A))$ is called the f-load of A.

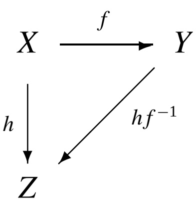
a). Let $f : X \to Y$ be a (surjective) quotient map. Then f is opne (respectively, closed) if and only if the f-load of each open (respectively, closed) in X is also open (respectively, closed in X.

b). Give an example of a surjective quotient map $f : X \to Y$ such that the f- load of an open subset of X need not be open.

8.5. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous surjective map.

a). Show that f is an quotient map if and only if for each topological space Z and each map $g : Y \to Z$, the continuity of $g \circ f$ implies the continuity of g.

b). Suppose that f is a quotient map and that $h : X \to Z$ is a continuous map. Assume that hf^{-1} is single-valued, i.e. h is constant on each fibre $f^{-1}(y)$, $y \in Y$. Then the map $hf^{-1} : Y \to Z$ is continuous and the diagram



is commutative. Moreover, the map hf^{-1} is open (respectively, closed) if and only if $h(U)$ is open (respectively, closed) whenever U is a f-saturated open (respectively, closed) in X.

c). Suppose that f is a quotient map, Z be any set and that $g : Y \to Z$ is a surjective map. Then show that $g \circ f$ is a quotient map if and only if g is a quotient map.

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† **Waclaw Sierpinski (1882-1969)** was born on 14 March 1882 in Warsaw, Russian Empire (now Poland) and died on 21 Oct 1969 in Warsaw, Poland.

Waclaw Sierpinski's father was a doctor. He attended school in Warsaw where his talent for mathematics was quickly spotted by his first mathematics teacher. This was a period of Russian occupation of Poland and it was a difficult time for the gifted Sierpinski to be educated in Poland. The Russians had forced their language and culture on the Poles in sweeping changes to all secondary schools implemented between 1869 and 1874. The Russian aim was to keep illiteracy in Poland as high as possible, so they discouraged learning and the number of students fell. Despite the difficulties, Sierpinski entered the Department of Mathematics and Physics of the University of Warsaw in 1899. It would be more accurate to describe it as the Czar's University since this was the official name of the University which had become a Russian university in 1869. The lectures at the University were all in Russian and the staff were entirely Russian. It is not surprising therefore that it would be the work of a Russian mathematician, one of his teachers Voronoy, that first attracted Sierpinski.

In 1903 the Department of Mathematics and Physics offered a prize for the best essay from a student on Voronoy's contribution to number theory. Sierpinski was awarded the gold medal in the competition for his dissertation. He described the events : ... *I was awarded a gold medal by the university for work in a competition on the theory of numbers. It was my first scientific work. It was accepted for publication in the 'Izvestia' of Warsaw University. However, in the following year there was a strike to produce a boycott of Russian Schools in Poland and I did not want to have my first work printed in the Russian language and that is why I had it withdrawn from print in Warsaw's 'Izvestia'. That is why it was not printed until 1907 in the mathematical magazine 'The works of Mathematics and Physics' published by Samuel Dickstein.*

Fifty years after he graduated from the University of Warsaw Sierpinski looked back at the problems that he had as a Pole taking his degree at the time of the Russian occupation: ... *we had to attend a yearly lecture on the Russian language. ... Each of the students made it a point of honour to have the worst results in that subject. ... I did not answer a single question ... and I got an unsatisfactory mark. ... I passed all my examinations, then the lector suggested I should take a repeat examination, otherwise I would not be able to obtain the degree of a candidate for mathematical science. ... I refused him saying that this would be the first case at our University that someone having excellent marks in all subjects, having the dissertation accepted and a gold medal, would not obtain the degree of a candidate for mathematical science, but a lower degree, the degree of a 'real student' (strangely that was what the lower degree was called) because of one lower mark in the Russian language.*

Sierpinski was lucky for the lector changed the mark on his Russian language course to 'good' so that he could take his degree. As he says: The policeman was human. The results in the prize essay that Sierpinski wrote in 1904 were a major contribution to a famous problem on lattice points. Suppose R(r) denotes the number of points (m, n), m, n [belongs] Z contained in a circle centre O, radius r. There exists a constant C and a number k with |R(r) - pr2| < Crk.

Let d be the minimal value of k. Gauss proved in 1837 that d [lte] 1. Sierpinski's major contribution was to show that it was possible to improve the inequality to d [lte] 2/3. In 1913 Edmund Landau shortened Sierpinski's proof and described the result as profound. Let us digress for a moment to discuss some further work which flowed from this result of Sierpinski on what is often called the 'Gauss circle problem'. In 1915 Hardy and Landau proved that d > 1/2, while in 1923 van der Corput proved that d < 2/3. The following year Littlewood and Walfisz proved that d [lte] 37/56, this being improved to d [lte] 163/247 the following year. Slight further improvements were made by Vinogradov in 1932 and Titchmarsh in 1934. The best result I [EFR] know is d [lte] 7/11.

Sierpinski graduated in 1904 and worked for a while as a school teacher of mathematics and physics in a girls school in Warsaw. However when the school closed because of a strike, Sierpinski decided to go to Krakóv to study for his doctorate. At the Jagiellonian University in Krakóv he attended lectures by Zaremba on mathematics, studying in addition astronomy and philosophy. He received his doctorate and was appointed to the University of Lvov in 1908. In fact it was in 1907 that Sierpinski first became interested in set theory. It happened when he came across a theorem which stated that points in the plane could be specified with a single coordinate. He wrote to Banachiewicz, who was at Göttingen at the time, asking him how such a result was possible. He received a one word reply 'Cantor'. Sierpinski began to study set theory and in 1909 he gave the first ever lecture course devoted entirely to set theory.

Throughout his life Sierpinski maintained an incredible output of research papers and books. During the years 1908 to 1914, when he taught at the University of Lvov, he published three books in addition to many research papers. These books were The theory of irrational numbers (1910), Outline of Set Theory (1912) and The theory of numbers (1912). When World War I began in 1914, Sierpinski and his family happened to be in Russia. At this time the governments of Austria and Russia tried to use the Polish question as a political weapon. Sierpinski was interned in Viatka. However Egorov and Luzin heard that he had been interned and arranged for him to be allowed to go to Moscow. Sierpinski spent the rest of the war years in Moscow working with Luzin . Together they began the study of analytic sets. In 1916, during his time in Moscow, Sierpinski gave the first example of an absolutely normal number, that is a number whose digits occur with equal frequency in whichever base it is written. Borel had proved such numbers exist but Sierpinski was the first to give an example. When World War I ended in 1918, Sierpinski returned to Lvov. However shortly after taking up his appointment again in Lvov he was offered a post at the University of Warsaw which he accepted. In 1919 he was promoted to professor at Warsaw and he spent the rest of his life there.

In 1920 Sierpinski, together with his former student Mazurkiewicz, founded the important mathematics journal Fundamenta Mathematicae. Sierpinski edited the journal which specialised in papers on set theory. From this period Sierpinski worked mostly in the area of set theory but also on point set topology and functions of a real variable. In set theory he made important contributions to the axiom of choice and to the continuum hypothesis . He studied the Sierpinski curve which describes a closed path which contains every interior point of a square. The length of the curve is infinity, while the area enclosed by it is 5/12 that of the square. Sierpinski continued to collaborate with Luzin on investigations of analytic and projective sets. His work on functions of a real variable include results on functional series, differentiability of functions and Baire's classification.

Sierpinski was also highly involved with the development of mathematics in Poland. He had been honoured with election to the Polish Academy in 1921 and he was made dean of the faculty at the University of Warsaw in the same year. In 1928 he became vice-chairman of the Warsaw Scientific Society and, in the same year was elected chairman of the Polish Mathematical Society.

In 1939 life in Warsaw changed dramatically with the advent of World War II. Sierpinski continued working in the 'Underground Warsaw University' while his official job was a clerk in the council offices in Warsaw. His publications continued since he managed to send papers to Italy. Each of these papers ended with the words: *The proofs of these theorems will appear in the publication of Fundamenta Mathematicae*, which everyone understood meant 'Poland will survive'. After the uprising of 1944 the Nazis burned his house destroying his library and personal letters. Sierpinski spoke of the tragic events of the war during a lecture he gave at the Jagiellonian University in Krakóv in 1945 (see [13]). He spoke of his students who had died in the war:

In July 1941 one of my oldest students Stanislaw Ruziewicz was murdered. He was a retired professor of Jan Kazimierz University in Lvov ... an outstanding mathematician and an excellent teacher. In 1943 one of my most distinguished students Stanislaw Saks was murdered. He was an assistant professor at Warsaw University, one of the leading experts in the world in the theory of the integral... In 1942 another student of mine, Adolf Lindenbaum was murdered. He was an assistant professor at Warsaw University and a distinguished author of works on set theory. After listing colleagues who were murdered in the war such as Schauder and others who died as a result of the war such as Dickstein and Zaremba, Sierpinski continued:

Thus more than half of the mathematicians who lectured in our academic schools were killed. It was a great loss for Polish mathematics which was developing favourably in some fields such as set theory and topology ... In addition to the lamented personal losses Polish mathematics suffered because of German barbarity during the war, it also suffered material losses. They burned down Warsaw University Library which contained several thousand volumes, magazines, mathematical books and thousands of reprints of mathematical works by different authors. Nearly all the editions of Fundamenta Mathematicae (32 volumes) and ten volumes of Mathematical Monograph were completely burned. Private libraries of all the four professors of mathematics from Warsaw University and also quite a number of manuscripts of their works and handbooks written during the war were burnt too.

Sierpinski was the author of the incredible number of 724 papers and 50 books. He retired in 1960 as professor at the University of Warsaw but he continued to give a seminar on the theory of numbers at the Polish Academy of Sciences up to 1967. He also continued his editorial work, as editor-in-chief of Acta Arithmetica which he began in 1958, and as an editorial board member of Rendiconti del Circolo Matematico di Palermo, Compositio Mathematica and Zentralblatt für Mathematik.

He received so many honours that it would be impossible to mention them all here. We list a few. He was awarded honorary degrees from the universities Lvov (1929), St Marks of Lima (1930), Amsterdam (1931), Tarta (1931), Sofia (1939), Prague (1947), Wroclaw (1947), Lucknow (1949), and Lomonosov University of Moscow (1967).

He was elected to the Geographic Society of Lima (1931), the Royal Scientific Society of Liège (1934), the Bulgarian Academy of Sciences (1936), the National Academy of Lima (1939), the Royal Society of Sciences of Naples (1939), the Accademia dei Lincei of Rome (1947), the German Academy of Science (1950), the American Academy of Arts and Sciences (1959), the Paris Academy (1960), the Royal Dutch Academy (1961), the Academy of Science of Brussels (1961), the London Mathematical Society (1964), the Romanian Academy (1965) and the Papal Academy of Sciences (1967).

Rotkiewicz, who was a student of Sierpinski's wrote : *Sierpinski had exceptionally good health and a cheerful nature. ... He could work under any conditions. ... He did not like any corrections to his papers. When someone suggested a correction he added a line to it: 'Mr X remarked that ...' He was a creative mind and liked creative mathematics. He was the greatest and most productive of Polish mathematicians.*