MA-231 Topology

12. Complete Metric Spaces[1])

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(1874-1932)

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(1892-1945)

12.1. (Examples of Complete metric spaces) **a).** Every finite dimensional norm linear space over $\mathbb{K}$ is complete. In particular, $\mathbb{K}^n$ (with the standard metric) is complete.

b). Let (X, d) be a complete metric space. Then show that the metric d^* on X defined by $d^*(x, y) := \min\{1, d(x, y)\}$ is also complete.

c). For a norm $\|-\|$ on $\mathbb{R}^n$, the map $x \mapsto x/(1+\|x\|)$ is a homeomorphism from $\mathbb{R}^n$ onto the open ball $\mathrm{B}(0\,;1)$. For $n \geq 1$, $\mathbb{R}^n$ is complete, but $\mathrm{B}(0\,;1)$ is not complete. (**Remark:** Therefore, the completeness of a metric space depends on the metric itself and not only on the topology. — In fact, any open subset of a complete metric space is homeomorphic to a complete metric space. For, let U be an open subset of a complete metric space (X, d). Then use the metric $d_1(x, y) = d(x, y) + \left|\frac{1}{d(x,(X\setminus U))} - \frac{1}{d(y,(X\setminus U))}\right|$.)

12.2. (Totally bounded metric spaces) A metric space (X, d) is called totally bounded[2]) if for each $\varepsilon > 0$, a finite number of ε-balls $\mathrm{B}(x_1; \varepsilon), \ldots, \mathrm{B}(x_r; \varepsilon)$ will cover X.

a). Every totally bounded metric space is bounded. The converse fails.

b). Every totally bounded metric space is seperable. In particular, every compact metric space is seperable.

c). (Equivalent conditions for totally boundedness) For a metric space (X, d), the following statements are equivalent: (i) The completion of (X, d) is compact. (ii) Every sequence has a Cauchy subsequence. (iii) For every $\varepsilon > 0$, there exists a finite subset $\{x_1, \ldots, x_r\}$ of X such that ε-balls $\mathrm{B}(x_1; \varepsilon), \ldots, \mathrm{B}(x_r; \varepsilon)$ cover X. (iv) For every $\varepsilon > 0$, there exists a finite subsets $A_1, \ldots A_r\}$ each of diameter $< \varepsilon$ such that $X = \cup_{i=1}^r A_i$. (v) Any infinite subset of X contains an infinite subset of diameter $< \varepsilon$.

12.3. (Equivalent conditions for completeness) For a metric space (X, d), the following statements are equivalent: (i) (X, d) is complete. (ii) For any descending sequence $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ of non-empty closed subsets with *diameters*[3]) is tending to 0, i.e. $\operatorname{diam}(A_n) \to 0$, the intersection $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$. (iii) Each infinite totally bounded subset has a cluster point. (**Remarks:** (1) In general, if $\operatorname{diam}(A_n) \to 0$, then either $\cap_{n\in\mathbb{N}} A_n = \emptyset$ or contains exactly one point. (2) If in (ii) we drop the assumption $\operatorname{diam}(A_n) \to 0$, we get a stronger property, namely compactness. — In the metric space $\mathbb{R}$, give ab example of descending sequence of non-empty closed subsets with empty intersection.)

12.4. Let (X, d) be a metric space.

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[1]) Complete metric spaces were introduced along with the definition of metric spaces by FRECHET. The proof that every metric space has a completion is based on the familiar method of defining the irrational numbers by means of cauchy sequences and is due to HAUSDORFF. Totaly bounded metric spaces were introduced by HAUSDORFF.

[2]) J. DIEUDONNE in [Foundations of Modern Analysis] used the word "precompact" to denote what we have referred to as totally bounded.

[3]) The diameter of a subset A of a metric space (X, d), $\operatorname{diam}(A)$, is defined to be the $\sup_{a,b\in A} d(a, b)$.

a). Suppose that there exists a $\varepsilon > 0$ such that all balls $\overline{\mathrm{B}}(x\,;\varepsilon)$, $x \in X$, are compact. Then (X, d) is complete. (**Hint:** The elements of any Cauchy-sequence will eventually lie in some Ball $\overline{B}(x;\varepsilon)$, and therefore have a cluster point in this compact ball, hence converge to this point. — **Remark:** But in general, a locally compact metric space is not complete.)

b). Suppose that Y is a dense subset of X and every Cauchy-sequence in Y converges to a point in X. Prove that (X, d) is complete.

c). If every countable closed subset of X is complete, then prove that (X, d) is complete.

d). If every closed ball of X is complete, then prove that (X, d) is complete.

e). If A and B are complete subsets of X, then prove that $A \cup B$ and $A \cap B$ are complete.

f). Let X_i, $i \in I$, be a finite family of non-empty metric spaces. Then the product (metric) space $\prod_i X_i$ is complete if and only if every metric space X_i is complete. The same assertion also holds in the case $I = \mathbb{N}$ is countably infinite. — (Recall that $d\left((x_n), (y_n)\right) := \sum_{n=0}^{\infty} \frac{1}{2^n} \min\left(1, d_n(x_n, y_n)\right)$ is a metric on the product $\prod_{n\in\mathbb{N}} X_n$.)

g). For a metric space (X, d), the following statements are equivalent: (i) Every Cauchy sequence in X is eventually constant. (ii) (X, d) is complete and discrete. (iii) Every subset of X is complete.

12.5. Let (X, d) be a metric space. Then

a). For a fixed $a \in X$, is the map $d(a, -) : X \to \mathbb{R}$, $y \mapsto d(a, x)$ uniformly continuous? For a fixed subset $A \subseteq X$, is the map $d(A, -) : X \to \mathbb{R}$, $y \mapsto d(A, x)$ uniformly continuous?

b). Let Y be another metric space. Assume that X is complete and that there exists a bijective continuous map $f : X \to Y$. If f^{-1} is uniformly continuous, then prove that Y is complete.

12.6. (Fixed Points) Let X be any set and let $f, g : X \to X$ be commuting maps, i.e. $f \circ g = g \circ f$. Then

a). The set of fixed points of g is invariant under g, i.e. $g(\mathrm{Fix}_X(f)) \subseteq \mathrm{Fix}_X(f)$.

b). Suppose that some power f^k of f has a unique fixed point x. Then Show that x is the only fixed point of f.

c). Suppose that X is a complete metric space and that some power f^k of f is a strict contraction (f need not be continuous). Then show that f has a unique fixed point.

d). Suppose that X is a complete metric space and that f is a strict contraction (g need not be continuous). Then show that there exists a unique common fixed point for f and g.

12.7. Let $X \neq \emptyset$ be a compact metric space and let $f : X \to X$ be a contraction map from X into itself. Then f has a unique fixed point x. For every initial value $x_0 \in X$, the sequence $x_n = f^n(x_0)$, $n \in \mathbb{N}$, converges to the fixed point x of f. (**Hint:** Consider a point in X at which the function $x \mapsto d\big(x, f(x)\big)$ attain its minimum.)

12.8. Let (X, d) be a complete metric space and let $f_m : X \to X$, $m = 1, 2 \ldots,$, $f : X \to X$ be strict contractions on X. Suppose that f_m, $m \in \mathbb{N}^+$ converges uniformly to f and that x_m, x be the fixed points of f_m, $m = 1, 2 \ldots,$, f, respectively. Then prove that x_m, $m \in \mathbb{N}^+$ converges to x in X.

12.9. (Extending Maps) Let $f' : X' \to Y$ be a *uniformly* continuous map on the dense subset X' of a metric space X into a complete metric space Y. Then there exists a (uniquely determined) continuous map $f : X \to Y$ such that $f|X' = f'$. This extension f of f' is also uniformly continuous. In particular, f' can be extended to the completion of X'.

12.10. Let (X, d) be a complete metric space.

a). Every sequence x_n, $n \in \mathbb{N}$, in X with $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$ is a Cauchy-sequence. For every $x := \lim x_n$ and every $n \in \mathbb{N}$, show that $d(x_n, x) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1})$.

b). (Generalisation of Banach's fixed point theorem) For a map $f : X \to X$, suppose that the iterations $f^n : X \to X$ of f has a *Lipschitz-constant*[4]) L_n, $n \in \mathbb{N}$. Suppose that $M := \sum_{n=0}^{\infty} L_n < \infty$. Then f has a unique fixed point x, and for every point $x_0 \in X$, the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$, is convergent and $d(x_n, x) \leq \left(\sum_{i=n}^{\infty} L_i\right) d(x_0, x_1)$ resp. $d(x_n, x) \leq M\, d(x_n, x_{n+1}) \leq L_n M\, d(x_0, x_1)$.

12.11. (Equivalent forms of Baire's theorem [5])) Let (X, d) be a complete metric space.

a). If $\{A_n \mid n \in \mathbb{N}\}$ is a sequence of nowhere dense subsets in X, then $X \setminus \cup_{n\in\mathbb{N}} A_n \neq \emptyset$.

b). If $\{A_n \mid n \in \mathbb{N}\}$ is a sequence of subsets in X with $X = \cup_{n\in\mathbb{N}} A_n$, then $\left(\overline{A_n}\right)^{\circ} \neq \emptyset$ for some $n \in \mathbb{N}$.

12.12. Let X be either a locally compact or complete metric space $\neq \emptyset$.

a). If X has no discrete points, then X is uncountable. (**Hint:** Baire's theorem.)

b). If X is countable, then X is totally disconnected. (**Remark:** There exists countably infinite connected Hausdorff topological space. Example?)

12.13. (Application of Baire's theorem) There does not exists a bijective continuous map $f : \mathbb{R} \to \mathbb{R}^n$, $n \geq 2$. (**Hint:** Such a map f induces a homeomorphism $I \to f(I)$ for every compact interval I. Now Baire's theorem $f([-n, n])$ for some n contains a compact ball $\overline{\mathrm{B}}(x, \varepsilon)$ with $\varepsilon > 0$. The inverse of f, restricted to this ball, is a continous injective mapping into $\mathbb{R}$, which is impossible already for circles, see Exercise 10.??. — **Remark:** For all $n > m \geq 1$ there exists a surjective continuous map $\mathbb{R}^m \to \mathbb{R}^n$. More generally, for $n \neq m$, there does not exists a bijective continuous map from $\mathbb{R}^m$ onto $\mathbb{R}^n$. This follows from the famous theorem of BORSUK-ULAM.)

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†† **René-Louis Baire (1874-1932)** René-Louis Baire was born on 21 Jan 1874 in Paris, France and died on 5 July 1932 in Chambéry, France. René Baire's father was a tailor and René was one of three children from the poor working class family who had to struggle under difficult financial circumstances. René grew up in Paris at the time when the Eiffel tower was being constructed. In 1886, when he was twelve years old, René won a scholarship to enable him to have a good education despite his family's poverty. He entered the Lycée Lakanal where he boarded and he became an outstanding student. He won two honourable mentions in the Concours Général, a competition between the top pupils from all the Lycées across France.

In 1890 René completed the advanced classes at the Lycée Lakanal and entered the special mathematics section of the Lycée Henri IV. After completing one year of preparation at this Lycée, he passed the entrance examinations for both the École Polytechnique and the École Normale Supérieure. He chose the latter as the place to study. Costabel writes ... *during his three years there [he] attracted attention by his intellectual maturity. He was a quiet young man who kept to himself and was profoundly introspective. During this period he was found to be in delicate health.*

At the École Normale Supérieure Baire attended lectures by Jules Tannery and Goursat and, in addition, he attended lectures by Hermite, Emile Picard and Poincaré at the Sorbonne. While he was a student he assisted with the editing of Poincaré's lectures, which he attended in 1894, on the propagation of heat. Having received his licentiate Baire proceeded toward his "agregation" but, although he was the best student in the written parts of this examination, he was only third overall after the oral examination.

His poor performance in the oral is worth describing in more detail since it was to have a great affect on the future direction of Baire's research. He was asked to prove the continuity of the exponential function but when he was in the middle of the proof he realised that: ... *his demonstration of continuity, which he had learnt at the Lycée Henri IV, was purely an artifice, since it did not refer sufficiently to the definition of the function.*

The examiners were hard on Baire and he was extremely disappointed with the outcome, but he then determined to examine again his analysis course while researching into the concept of continuity of a general function. However, the immediate result of his passing his agregation was that he obtained his first post as a professor at a lycée. An appointment in Bar- le-Duc gave him reasonable financial security but he was unhappy that living in Bar-le-Duc meant that he had no opportunity for close contacts with university life.

At the Lycée in Bar-le-Duc Baire worked on the theory of functions and the concept of a limit. Around this time he discovered conditions under which a function is a limit of a sequence of continuous functions. Shortly after this Baire set up his classification of functions. Class 1 functions were those functions which were the limit of a sequence of continuous functions. Class 2 functions were those functions which were the limit of a sequence of Class 1 functions, while Class 3 functions were those functions which were the limit of a sequence of Class 2 functions.

Baire was awarded a scholarship to allow him to continue his studies in Italy and there he met and established a close friendship with Volterra. While he worked in the lycée, Baire wrote a doctoral thesis on discontinuous functions. He was examined on 24 March 1899 by a board consisting of Darboux, Appell and Emile Picard, and they awarded him the doctorate. However: *The few objections, which Baire fully appreciated, proved that he had embarked on a new road and would not find it easy to convince his listeners.*

Even before presenting his thesis Baire had suffered from poor health and, after the award of his doctorate, he was only able to contribute to mathematics for a few short spells. He continued to teach in lycées (he taught in Troyes, Bar-le-duc and Nancy) but was not happy teaching at this low level. In 1901 Baire was appointed to the University of Montpellier as a "Maitre des conferences". This post saw him preparing students for the "agregation" examination, a position he enjoyed much more than teaching in lycées. While at Montpellier he wrote a paper on irrational numbers and limits.

In 1904 he was awarded a Peccot Foundation Fellowship which was to allow young school teachers to spend a semester in a university developing their skills. Baire spent the semester at the Collège de France where he lectured on the subject of his thesis and had the lectures published the next year. Baire returned to Montpellier where he suffered the first severe attack of illness but after a while the worst of the attack passed and he was able to work again. He was appointed to a university post in 1905 when he joined the Faculty of Science at Dijon. In 1907 he was promoted to Professor of Analysis at Dijon.

Baire's health had never been good since he was young but from the time he was at the Lycée at Bar-le-Duc it began to deteriorate to the stage that it prevented him from working. The bad spells became more frequent, immobilising him for long periods. Apart from problems with his oesophagus that had plagued him since his youth, he developed a kind of psychological disorder which, using his own description, "debilitated" him occasionally. Apparently he eventually became unable to undertake work which required him to concentrate, and research in mathematics became impossible at these times. Between 1909 and 1914 he continued trying to undertake his teaching duties, but this became more and more difficult. Then near the beginning of 1914 he requested leave so that he might try to recover his health.

Baire went first to Alésia, then he went to Lausanne. It was while he was in Lausanne that World War I began and he was not able to return to France. He spent the war years from 1914 until 1918 in Lausanne in quite difficult financial circumstances.

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[4]) i.e. $d(f^n(x), f^n(y)) \leq L_n d(x, y)$ for all $x, y \in X$.

[5]) Note that a need for Baire's theorem crops up from time to time and when this need arises, Baire's theorem is an indispensable tool. There is some rather undescriptive terminology whihc is often used in connection with Baire's theorem. A subset of a metrci space is called a set of the first category if it can be represented as the union of sequence of nowhere dense subsets, and a set of second category if it is not a set of first category. Baire's theorem — sometimes called the *Baire's category theorem* — can now be expressed as: *any complete metric space is a set of the second category.*

It is interesting to consider the various causes suggested by his contemporaries to account for his illness. Some suggested that the cause for his problems lay in intellectual overexertion in his student days. His close family, and others close to him, blamed his illness on his deep feelings of frustration that his achievements were not being recognised by the academic authorities. Baire felt that he deserved a professorship in Paris and failing to achieve this, it was suggested, caused him depression and hence his ill health.

Certainly Baire felt that men such as Lebesgue, who was younger than Baire, had been unfairly preferred to him. He first fell out with Lebesgue in 1904, when he taught his course at the Collège de France, over who had the most right to teach such a course. Their rivalry turned into a more serious argument later in Baire's life. Baire also fell out with de la Vallée Poussin which may be surprising to those who know that Baire's ideas entered the mainstream of mathematics through de la Vallée Poussin's well-known treatise. The letters written to Baire by de la Vallée Poussin, and reproduced to give an idea of the reasons for their arguments which seem to centre round the fact that de la Vallée Poussin had classified by order of importance mathematical discoveries of Lebesgue and Baire.

While on the topic of letters, we should remark that there are fifty letters written by Baire to Emile Borel. The first five are written during 1898 beginning during the time that Baire was in Italy. There is a gap from the fifth letter, dated 22 May 1898 to the sixth dated 4 February 4 1902. The gap is explained by Baire's first serious illness over the period he taught at Bar-le-Duc. In the letters reproduced, Baire writes in great detail about his research ideas, including the Baire classification of functions, sets of first and second category, and semicontinuity. In the letters he also discusses Cantor's set theory and the foundations of mathematics.

It appears that it was not only Baire's family who felt he had been hard done by, for after 1918 many in the mathematical community seemed to be trying to make amends for his lack of recognition. In 1918 some suggested that a chair at the Collège de France, which he undoubtedly deserved, would lift Baire's depression, helping him to regain his intellectual vigour, but apparently these suggestions never materialised. Unable to resume his duties, Baire lived on the shores of Lake Geneva and he was there when he received the Chevalier de la Legion d'Honneur and in 1922 when he was elected to the Académie des Sciences. He retired in 1925 and spent his last years in the solitude of hotel rooms on the shores of the Lake of Leman. Although he received a reasonable pension, inflation over these years soon meant that he ended his life with financial difficulties similar to those of his youth.

Despite being unable to work for long periods, Baire wrote a number of important analysis books including Théorie des nombres irrationels, des limites et de la continuité (1905) and Leçons sur les théories générales de l'analyse, 2 Vols. (1907-8). Baire made a decisive step in moving away from the intuitive idea of functions and continuity and he saw clearly that a theory of infinite sets was fundamental for rigorous real analysis. He wrote in his doctoral thesis: *Generally speaking, in the framework of ideas that here concern us, every problem in the theory of functions leads to certain questions in the theory of sets, and it is to the degree that these latter questions are resolved, that it is possible to solve the given problem more or less completely.* When his health was good, the quality of his lectures received rather differing opinions from his students. Some described his lectures as very clear, but others claimed that what he taught was so difficult that it was beyond human ability to understand. Baire, aware of these comments, wrote: *... but look at Denjoy - he understood it, hence it must not be so difficult ...*

Denjoy, who was Baire's most famous student, certainly understood Baire's ideas and developed them in his own work. He wrote that Baire was: *.. not an agreeable character ... [and] not a person of enormous culture ...[but] constantly tormented due to the fatigue of his brain.* On the other hand Denjoy described him as: *... an excellent person.* Another of his students, Reault, wrote much more positively describing Baire as having: *... paternal concern ... [He had] high intellectual qualities ... a penetrating mathematical mind [with] the extent and depth of his knowledge ... [and] the greatness of his character.*

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†† **Stefan Banach (1892-1945)** was born on 30 March 1892 in Kraków, Austria-Hungary (now Poland) and died on 31 Aug 1945 in Lvov, (now Ukraine). Stefan Banach's father was Stefan Greczek. The first thing to notice is that Banach was not his father's surname, but Banach was given his father's first name. Stefan Greczek was a tax official who was not married to Banach's mother who vanished from the scene after Stefan was baptised, when he was only four days old, and nothing more is known of her. The name given as Stefan's mother on his birth certificate is Katarzyna Banach. She is thought by some to have been the servant of Stefan's mother, while others claim that she was a laundress who took care of Stefan when he was very young. In later life Banach tried to find out who his mother was but his father refused to say anything except that he had been sworn to secrecy over her identity.

Stefan Greczek was born in a small village called Ostrowsko, some 50 km south of Kraków. It was to Ostrowsko, to his grandmother's home, that Banach was taken after his baptism. However, when Banach's grandmother took ill, Stefan Greczek arranged for his son to be brought up by Franciszka Plowa who lived in Kraków with her daughter Maria. Although Banach never went back to live with his grandmother, he did visit her frequently as he grew up. Maria's guardian was a French intellectual Juliusz Mien and he quickly recognised the talents that Banach had. Mien taught the young boy to speak French and in general gave him an appreciation for education.

Banach attended primary school in Kraków, leaving the school in 1902 to begin his secondary education at the Henryk Sienkiewicz Gymnasium No 4 in Kraków. By a fortunate coincidence, one of the students in Banach's class was Witold Wilkosz who himself went on to become a professor of mathematics. The school does not appear to have been a particularly good one and in 1906 Wilkosz left to move to a better Gymnasium. Banach, however, remained at Henryk Sienkiewicz Gymnasium No 4 although he maintained contact with Wilkosz.

During his first few years at the Gymnasium Banach achieved first class grades with mathematics and natural sciences being his best subjects. A fellow school pupil recalled Banach at this period in his life : [Banach] was pleasant in dealings with his colleagues, but outside of mathematics he was not interested in anything. If he spoke at all, he would speak very rapidly, as rapidly as he thought mathematically. ... Wilkosz was a similar phenomenon. Between the two of them there was no mathematical problem that they could not speedily tackle. Also, while Banach was faster in mathematical problems, Wilkosz was phenomenally fast in solving problems in physics, which were of no interest to Banach.

The excellent grades of his early years gave way to poorer grades as he approached his final school examination. He passed this examination in 1910 but he failed to achieve a pass with distinction, an honour which went to about one quarter of the students. On leaving school Banach and Wilkosz both wanted to study mathematics, but both felt that nothing new could be discovered in mathematics so each chose to work in a subject other than mathematics. Banach chose to study engineering, Wilkosz chose oriental languages. That two such outstanding future mathematicians could make a decision for this reason must mean that there was nobody to properly advise them.

Banach's father had nver given his son much support, but now once he left school he quite openly told Banach that he was now on his own. Banach left Kraków and went to Lvov where he enrolled in the Faculty of Engineering at Lvov Technical University. It is almost certain that Banach, without any financial support, had to support himself by tutoring. This must have occupied quite a lot of his time and when he graduated in 1914 he had taken longer to complete the course than was normal. He had returned to Kraków frequently during the period of his studies in Lvov from 1910 to 1914. It is not entirely clear what Banach's plans were in 1914 but the outbreak of World War I in August, shortly after his graduation, saw Banach leave Lvov.

Lvov was, at the time Banach studied there, under Austrian control as it had been from the partition of Poland in 1772. In Banach's youth Poland, in some sense, did not exist and Russia controlled much of the country. Warsaw only had a Russian language university and was situated in what was named "Vistula Land". With the outbreak of World War I, the Russian troops occupied the city of Lvov. Banach was not physically fit for army service, having poor vision in his left eye. During the war he worked building roads but also spent time in Kraków where he earned money by teaching in the local schools. He also attended mathematics lectures at the Jagiellonian University in Kraków and, although this is not completely certain, it is believed that he attended Zaremba's lectures.

A chance event occurred in the spring of 1916 which was to have a major impact on Banach's life. Steinhaus, who had been undertaking military service, was about to take up a post at the Jan Kazimierz University in Lvov. However he was living in Kraków in the spring of 1916, waiting to take up the appointment. He would walk through the streets of Kraków in the evenings and, as he related in his memoirs: During one such walk I overheard the words "Lebesgue measure". I approached the park bench and introduced myself to the two young apprentices of mathematics. They told me they had another companion by the name of Witold Wilkosz, whom they extravagantly praised. The youngsters were Stefan Banach and Otto Nikodym. From then on we would meet on a regular basis, and ... we decided to establish a mathematical society.

Steinhaus told Banach of a problem which he was working on without success. After a few days Banach had the main idea for the required counterexample and Steinhaus and Banach wrote a joint paper, which they presented to Zaremba for publication. The war delayed publication but the paper, Banach's first, appeared in the Bulletin of the Kraków Academy in 1918. From the time that he produced these first results with Steinhaus, Banach started to produce important mathematics papers at a rapid rate. Of course it is impossible to say whether, without the chance meeting with Steinhaus, Banach would have followed the route of research in mathematics. It was also through Steinhaus that Banach met his future wife Lucja Braus. They were married in the mountain resort of Zakopane in 1920. On Steinhaus's initiative, the Mathematical Society of Kraków was set up in 1919. Zaremba chaired the inaugural meeting and was elected as the first President of the Society. Banach lectured to the Society twice during 1919 and continued to produce top quality research papers. The Mathematical Society of Kraków went on to became the Polish Mathematical Society in 1920.

Banach was offered an assistantship to Lomnicki at Lvov Technical University in 1920. He lectured there in mathematics and submitted a dissertation for his doctorate under Lomnicki's supervision. This was, of course, not the standard route to a doctorate, for Banach had no university mathematics qualifications. However, an exception was made to allow him to submit On Operations on Abstract Sets and their Application to Integral Equations. This thesis *... is sometimes said to mark the birth of functional analysis.*

In 1922 the Jan Kazimierz University in Lvov awarded Banach his habilitation for a thesis on measure theory. The University Calendar for 1921-22 reports: *On 7 April 1922, by resolution of the Faculty Council, Dr Stefan Banach received his habilitation for a Docent of Mathematics degree. He was appointed Professor Extraordinary of that subject by decree of the Head of State issued on 22 July 1922.*

In 1924 Banach was promoted to full professor and he spent the academic year 1924-25 in Paris. The years between the wars were extremely busy one for Banach. As well as continuing to produce a stream of important papers, he wrote arithmetic, geometry and algebra texts for high schools. He also was very much involved with the publication of mathematics. In 1929, together with Steinhaus, he started a new journal Studia Mathematica and Banach and Steinhaus became the first editors. The editorial policy was: *... to focus on research in functional analysis and related topics.*

Another important publishing venture, begun in 1931, was a new series of Mathematical Monographs. These were set up under the editorship of Banach and Steinhaus from Lvov and Knaster, Kuratowski, Mazurkiewicz, and Sierpinski from Warsaw. The first volume in the series Théorie des Opérations linéaires was written by Banach and appeared in 1932. It was a French version of a volume he originally published in Polish in 1931 and quickly became a classic. In 1936 Banach gave a plenary address at the International Congress of Mathematicians in Oslo. In this address he described the work of the whole of the Lvov school, and he also spoke of the plans which they had to develop their ideas further. Another important influence on Banach was the fact that Kuratowski was appointed to the Lvov Technical University in 1927 and worked there until 1934. Banach collaborated with Kuratowski and they wrote some joint papers during this period.

The way that Banach worked was unconventional. He liked to do mathematical with his colleagues in the cafés of Lvov. Ulam recalls frequent sessions in the Scottish Café: *It was difficult to outlast or outdrink Banach during these sessions. We discussed problems proposed right there, often with no solution evident even after several hours of thinking. The next day Banach was likely to appear with several small sheets of paper containing outlines of proofs he had completed.*

Andrzej Turowicz, also a professor of mathematics at the an Kazimierz University in Lvov, also described Banach's style of working: *[Banach] would spend most of his days in cafés, not only in the company of others but also by himself. He liked the noise and the music. They did not prevent him from concentrating and thinking. There were cases when, after the cafés closed for the night, he would walk over to the railway station where the cafeteria was open around the clock. There, over a glass of beer, he would think about his problems.*

In 1939, just before the start of World War II, Banach was elected as President of the Polish Mathematical Society. At the beginning of the war Soviet troops occupied Lvov. Banach had been on good terms with the Soviet mathematicians before the war started, visiting Moscow several times, and he was treated well by the new Soviet administration. He was allowed to continue to hold his chair at the university and he became the Dean of the Faculty of Science at the university, now renamed the Ivan Franko University. Banach's father came to Lvov fleeing from the German armies advancing towards Kraków. Life at this stage was little changed for Banach who continued his research, his textbook writing, lecturing and sessions in the cafés. Sobolev and Aleksandrov visited Banach in Lvov in 1940, while Banach attended conferences in the Soviet Union. He was in Kiev when Germany invaded the Soviet Union and he returned immediately to his family in Lvov.

The Nazi occupation of Lvov in June 1941 meant that Banach lived under very difficult conditions. He was arrested under suspicion of trafficking in German currency but released after a few weeks. He survived a period when Polish academics were murdered, his doctoral supervisor Lomnicki dying on the tragic night of 3 July 1941 when many massacres occurred. Towards the end of 1941 Banach worked feeding lice in German institute dealing with infectious diseases. Feeding lice was to be his life during the remainder of the Nazi occupation of Lvov up to July 1944. As soon as the Soviet troops retook Lvov Banach renewed his contacts. He met Sobolev outside Moscow but clearly he was by this time seriously ill. Sobolev, giving an address at a memorial conference for Banach, said of this meeting for example: *Despite heavy traces of the war years under German occupation, and despite the grave illness that was undercutting his strength, Banach's eyes were still lively. He remained the same sociable, cheerful, and extraordinarily well-meaning and charming Stefan Banach whom I had seen in Lvov before the war. That is how he remains in my memory: with a great sense of humour, an energetic human being, a beautiful soul, and a great talent.*

Banach planned to go to Kraków after the war to take up the chair of mathematics at the Jagiellonian University but he died in Lvov in 1945 of lung cancer. Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces. In addition, he contributed to measure theory, integration, the theory of sets, and orthogonal series. In his dissertation, written in 1920, he defined axiomatically what today is called a Banach space. The idea was introduced by others at about the same time, for example Wiener introduced the notion but did not develop the theory. The name 'Banach space' was coined by Fréchet. Banach algebras were also named after him. A Banach space is a real or complex normed vector space that is complete as a metric space under the metric $d(x,y)=||x-y||$ induced by the norm. The completeness is important as this means that Cauchy sequences in Banach spaces converge. A Banach algebra is a Banach space where the norm satisfies $||xy|| \leq ||x||.||y||$.

The importance of Banach's contribution is that he developed a systematic theory of functional analysis, where before there had only been isolated results which were later seen to fit into the new theory. The theory generalised the contributions made by Volterra, Fredholm and Hilbert on integral equations. Banach proved a number of fundamental results on normed linear spaces, and many important theorems are today named after him. There is the Hahn- Banach theorem on the extension of continuous linear functionals, the Banach-Steinhaus theorem on bounded families of mappings, the Banach- Alaoglu theorem, the Banach fixed point theorem and the Banach-Tarski paradoxical decomposition of a ball.

The Banach-Tarski paradox appeared in a joint paper of the two mathematicians in 1926 in Fundamenta Mathematicae entitled Sur la décomposition des ensembles de points en partiens respectivement congruent. The puzzling paradox shows that a ball can be divided up into subsets which can be fitted together to make two balls each identical to the first. The axiom of choice is needed to define the decomposition and the fact that it is able to give such a non-intuitive result has made some mathematicians question the use of the axiom. The Banach-Tarski paradox was a major contribution to the work being done on axiomatic set theory around this period.

Banach's open mapping of 1929 also uses set-theoretic concepts, this time concepts introduced by Baire in his 1899 dissertation.