# MA-231 Topology

## 13. Function spaces[1])

### — Uniform Convergence, Stone-Weierstrass theorem, Arzela-Ascoli theorem

November 22, 2004

**Karl Theodor Wilhelm Weierstrass**† **(1815-1897)**

**Marshall Harvey Stone**†† **(1903-1989)**

First recall the following definitions and results :

Our overall aim in this section is the study of the compactness and completeness properties of a subset $\mathcal{F}$ of the set $Y^X$ of all maps from a space $X$ into a space $Y$. To do this a usable topology must be introduced on $\mathcal{F}$ (presumably related to the structures of $X$ and $Y$) and when this is has been done $\mathcal{F}$ is called a function space.

**N13.1.** (The Topology of Pointwise Convergence [2])) Let $X$ be any set, $Y$ be any topological space and let $f_n : X \to Y$, $n \in \mathbb{N}$, be a sequence of maps. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent, if for every point $x \in X$, the sequence $f_n(x)$, $n \in \mathbb{N}$, in $Y$ is convergent.

**a).** The (Tychonoff) product topology on $Y^X$ is determined solely by the topology of $Y$ (even if $X$ is a topological space) the structure on $X$ plays no part. *A sequence $f_n : X \to Y$, $n \in \mathbb{N}$, in $Y^X$ converges to a function $f$ in $Y^X$ if and only if for every point $x \in X$, the sequence $f_n(x)$, $n \in \mathbb{N}$, in $Y$ is convergent.* This provides the reason for the name the topology of pointwise convergence; this topology is also simply called the pointwise topology and is denoted by $\mathcal{T}_{\text{ptc}}$. Suppose that $Y$ is a Hausdorff topological space, then the map

$$f = \lim_{n\to\infty} f_n \quad \text{with} \quad f(x) := \lim_{n\to\infty} f_n(x)$$

is called the limit map or the limit of the sequence $f_n$, $n \in \mathbb{N}$. Note that for each $x \in X$, $f(x)$ is uniquely determined, since $Y$ is Hausdorff.

**b).** (Compact subsets of in the topology of pointwise convergence) Let $X$ be a set and let $Y$ be a Hausdorff topological space and let $\mathcal{F} \subseteq Y^X$ with the topology of pointwise convergence. Then $\mathcal{F}$ is compact if and only (i) $\mathcal{F}$ is pointwise closed, i.e. $\mathcal{F}$ is closed in the topology of pointwise convergence on $Y^X$. (ii) For each $x \in X$, $\pi_x(\mathcal{F}) = \{f(x) \mid f \in \mathcal{F}\}$ has a compact closure in $Y$, where $\pi_x : Y^X \to Y$ is the $x$-th projection $f \mapsto f(x)$.

---

[1]) The study of sets or spaces of functions began with the work of ASCOLI in 1883 ["Le curve Limite di una Varietá Data di Curve," *Mem. Accad. Lincei* (3) **18**, 521-586 (1883)], ARZELA in 1889 ["Funzioni di Linee," *Atti della Reale Accademia dei Lincei, Rendiconti* , **5**, 342-348 (1889)] and HADAMARD in 1898. These papers mark the beginning , not only of function space theory, but of general topology itself, for it was the questions which they raised that men like FRECHET, RIESZ, WEYL and finally HAUSDORFF were trying to answer. Coherent attempts to study topologies on spaces of functions in their own right began in 1935 with TYCHONOFF, who pointed out that his product topology on $Y^X$ is just the topology of pointwise convergence. The term *function space* is used much earlier in connection with questions of a topological nature about sets of functions.

[2]) The study of pointwise convergence of (sequences of) functions is as old as the calculus. The study of uniform convergence began hard on the heels of the formalization of the notion of limit by CAUCHY. CAUCHY in 1821 published a faulty proof of the false statement that the pointwise limit of a sequence of continuous functions is always continuous. FOURIER and ABEL found counter examples in the context of Fourier series. DIRICHLET then analyzed Cauchy's proof and found the mistake: the notion of pointwise convergence had to be replaced by uniform convergence. In the last half of the 19th century, in the hands of HEINE, WEIERSTRASS, RIEMANN and others, uniform convergence came into its own in applications to integrations theory and Fourier seris.

**c).** Let $X$ be a set and let $Y$ be a Hausdorff topological space and let $\mathcal{F} \subseteq Y^X$. Let $\mathcal{T}$ be a topology on $\mathcal{F}$. If $\mathcal{T}$ is compact and larger than $\mathcal{T}_{\rm ptc}$, then the identity map $\mathrm{id} : (\mathcal{F}, \mathcal{T}) \to (\mathcal{F}, \mathcal{T}_{\rm ptc})$ is continuous and therefore a homeomorphism! (note that $(\mathcal{F}, \mathcal{T}_{\rm ptc})$ is Hasudorff, since $Y$ is).

compact topology

**N13.2.** (Uniform convergence) Let $X$ be any set, $(Y, d)$ be a metric space. Then in addition to the above pointwise convergence, we also define the uniform convergence : A sequence $f_n : X \to Y$, $n \in \mathbb{N}$, of maps from $X$ into $Y$ converges uniformly to a map $f : X \to Y$, if for every $\varepsilon > 0$, there exits a $n_0 \in \mathbb{N}$ such that $d\big(f(x), f_n(x)\big) \le \varepsilon$ for all $n \ge n_0$ and all $x \in X$.

**a).** The uniform convergence of $f_n : X \to Y$, $n \in \mathbb{N}$, implies the pointwise convergence. Moreover, if $Y$ is complete then we have the following Cauchy's criterion for the uniform convergence :

**b).** (Cauchy's criterion for uniform convergence) A seqeunce $(f_n)$ of functions $f_n : X \to Y$ from a set $X$ into a complete metric space $Y$ converges uniformly if and only if for every $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that for all $m, n \ge n_0$ and all $x \in X$, we have $d\big(f_m(x), f_n(x)\big) \le \varepsilon$.

**c).** Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a uniformly convergent sequence of continuous maps from the topological space $X$ into the metric space $Y$. Then the limit function $f = \lim f_n$ is also continuous. ( **Proof :** Let $a \in X$ and $\varepsilon > 0$ be given. Then there exits a $n \in \mathbb{N}$ and a nhood $U$ of $a$ such that $d\big(f(x), f_n(x)\big) \le \varepsilon/3$ for all $x \in X$ and $d\big(f_n(a), f_n(x)\big) \le \varepsilon/3$ for all $x \in U$. Then for $x \in U$, we have $d\big(f(a), f(x)\big) \le d\big(f(a), f_n(a)\big) + d\big(f_n(a), f_n(x)\big) + d\big(f_n(x), f(x)\big) \le \varepsilon$ .)

**d).** Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a *locally uniformly convergent* sequence of continous maps from the topological space $X$ in the metric space $Y$. Then the limit function $\lim f_n$ is also continuous. (**Hint :** Immediate from the fact that continuity is a local property. — Recall that a sequence $f_n : X \to Y$ locally uniformly convergent, if for every $x \in X$, there exists a nhood $U$ of $x$ in $X$ such that the sequence $f_n|U$, $n \in \mathbb{N}$, is uniformly convergent on $U$.)

**N13.3.** (The topology (or metric) of the uniform convergence on $Y^X$) Let $X$ be any set and let $(Y, d)$ be a metric space. Then on the set $Y^X$ of all maps from $X$ into $Y$, there is a natural metric defined by : $\rho(f, g) := \mathrm{Sup}\,\big\{ \mathrm{Min}\,\big(d\big(f(x), g(x)\big), 1\big) \bigm| x \in X \big\}$ for $f, g \in Y^X$.[3]) Then : *$\rho$ is a metric $Y^X$ and the sequence $f_n \in Y^X$, $n \in \mathbb{N}$, uniformly converges to a map $f \in Y^X$ if and only if the sequence $f_n \in Y^X$, $n \in \mathbb{N}$, in the metric space $Y^X$ converges to $f$.* Therefore the topology of $Y^X$ defined by the metric $\rho$ is called the topology (or metric) of the uniform convergence on $Y^X$ or just the uniform topology and is denoted by $\mathcal{T}_{\rm uc}$ .

**a).** Let $X$ be a topological space and let $Y$ be a metric space. Then the set $\mathrm{C}(X, Y)$ of all continuous maps from $X$ into $Y$ is a closed subset of $Y^X$. Further, if $Y$ is complete then $Y^X$ is also complete and in particular, the set $\mathrm{C}(X, Y)$ is a complete metric space with respect to the metric of uniform convergence. (**Hint :** Use 13.1-c) and the Cauchy's criterion for uniform convergence 13.1-b))

**b).** If $X$ is a compact topological space, then in this case we choose the natural distance between two continuous functions $f, g : X \to \mathbb{K}$ as the supremum norm or Tschebyschev norm defines a distance function $\|g - f\| = \|g - f\|_X = \mathrm{Sup}\,\big\{|g(x) - f(x)| \bigm| x \in X\big\}$. Moreover, if $X \neq \emptyset$, then there exists a $x_0 \in X$ such that $\|g - f\|_X = |g(x_0) - f(x_0)|$ .

In addition to the topology of pointwise convergence and the topology of uniform convergence, there are other interesting and useful topologies on $Y^X$ and $\mathrm{C}(X, Y)$, for example, the *topology of compact convergence* and the *compact-open topology*. We shall recall these below :

**N13.4.** (The topology of compact convergence on $Y^X$) Let $X$ be any topological space and let $(Y, d)$ be a metric space. For $f \in Y^X$, a real number $\varepsilon > 0$ and a compact subset $K$ of $X$, let $\mathrm{B}_K(f; \varepsilon) := \{g \in Y^X \mid \sup_{x \in K} d\,(f(x), g(x)) < \varepsilon\}$. The subsets $\{\mathrm{B}_K(f; \varepsilon) \mid f \in Y^X, \varepsilon > 0$ and a compact subset $K \subseteq X\}$ form a basis for a topology on $Y^X$; this topology is called the topology of compact convergence or the topology of uniform convergence on compact sets. The justification for the choice of this terminology comes from the following : *A sequence $f_n : X \to Y$, $n \in \mathbb{N}$, in $Y^X$ converges to a function $f$ in $Y^X$ in the topology of compact convergence if and only if for every compact subset $K$ of $X$, the sequence $f_n|K$, $n \in Y^K$ converges uniformly to $f|K$ in $Y^K$.*

---

[3]) The choice of Min $\big(d\big(f(x), g(x)\big), 1\big)$ gurantee that $\rho(f, g)$ is finite; this can also be achieved by assuming (see ???) that the metric $d$ on $Y$ is bounded. In the case when $X$ is a compact topological space and *continuous* maps $f, g : X \to Y$ it is immediate that $\rho(f, g)$ is finite. In both these cases therefore for $\rho(f, g)$ we will make the choice $\mathrm{Sup}\,\big\{d\big(f(x), g(x)\big) \bigm| x \in X\big\}$.

**N13.5.** The relation between the above three topologies is the following: $\mathcal{T}_{\rm uc} \supseteq \mathcal{T}_{\rm cc} \supseteq \mathcal{T}_{\rm ptc}$. Moreover, if $X$ is compact, then the first two coincise and if $X$ is discrete, then the second two coincide. (**Remark:** The definitions of the uniform topology and the compact convergence topology made specific use of the metric $d$ of the metric space $(Y, d)$. But the topology of pointwise convergence did not, in fact, it is defined for any topological space $Y$. It is natural to ask whether either of these other topologies can be extended to the case where $Y$ is an arbitrary topological space. *There is no satisfactory answer for this question for the space* $Y^X$. But for the subspace $\mathrm{C}(X, Y)$ of continuous functions from $X$ into $Y$, one can prove something. It turns out that there is in general a topology on $\mathrm{C}(X, Y)$, called the *compact-open topology* (see below), that coincides with the compact convergence topology when $Y$ is a mteric space. This topology is important in its own right.)

**N13.6.** (The compact-open topology[4]) on $Y^X$) Let $X$ and $Y$ be any topological spaces. For a compact subset $K \subseteq X$ and an open subset $U \subseteq Y$, let $(K, U) := \{f \in Y^X \mid f(K) \subseteq U\}$. the topology generated by the subsets $\{(K, U) \mid K$ compact subset of $X$ and $U$ an open subset of $Y\}$ is called the compact-open topology or $k$-topology on $Y^X$ and is denoted by $\mathcal{T}_{\rm co}$. In the case when $Y$ is a metric space, it is clear from the definition that the compact-open topology is finer than the topology of pointwise convergence, i.e. $\mathcal{T}_{\rm co} \supseteq \mathcal{T}_{\rm ptc}$.

**a).** Let $X$ be any topological space and let $(Y, d)$ be a metric space. Then the compact-open topology on $\mathrm{C}(X, Y)$ and the topology of compact convergence coincide, i.e. $\mathcal{T}_{\rm co} = \mathcal{T}_{\rm cc}$ on $\mathrm{C}(X, Y)$. In particular, The topology of compact convergence on $\mathrm{C}(X, Y)$ does not depend on the metric $d$ on $Y$. Therefore if $X$ is compact, the uniform topology on $\mathrm{C}(X, Y)$ does not depend on the metric $d$ on $Y$.

The fact that the definition of the compact-open topology does not involve a metric is just one of the useful features. Another is the fact that it satisfies the requirement of "joint continuity" – roughly speaking this means that the expression $f(x)$ is continuous not only in the single "variable" $x$, but it is continuous jointly in both the "variables" $x$ and $f$. More precisely:

**b).** *Let $X$ be a locally compact (Hausdorff) topological space and consider $\mathrm{C}(X, Y)$ with the compact-open topology. Then the evaluation map $e : X \times \mathrm{C}(X, Y) \to Y$ defined by $(x, f) \mapsto f(x)$ is continuous.*

**N13.7.** (Stone-Weierstrass theorems) For a compact topological space $X$ we consider the $\mathbb{K}$-algebra $\mathrm{C}_{\mathbb{K}}(X)$ of all continuous $\mathbb{K}$-valued functions on $X$ with the metric of uniform convergence. We proved the following generalisation of the classical aproximation theorem of Weierstrass:

(1) (Approximation theorem of Stone-Weierstrass for $\mathbb{R}$-valued functions) *Let $X$ be a compact topological space and let $A$ be a $\mathbb{R}$-subalgebra of the algebra $\mathrm{C}_{\mathbb{R}}(X)$ of all continuous real valued functions on $X$. If the algebra $A$ seperates points of $X$, i.e. for every two distinct points $x, y \in X$, there exists a function $f \in A$ with $f(x) \neq f(y)$. Then $A$ is dense in $\mathrm{C}_{\mathbb{R}}(X)$.*

(2) (Approximation theorem of Stone-Weierstrass for $\mathbb{C}$- valued functions) *Let $X$ be a compact topological space and let $A$ be a $\mathbb{C}$-subalgebra of the algebra $\mathrm{C}_{\mathbb{C}}(X)$ of all continuous real valued functions on $X$. Suppose that* (i) *the algebra $A$ seperates points of $X$, i.e. for every two distinct points $x, y \in X$, there exists a function $f \in A$ with $f(x) \neq f(y)$.* (ii) *If $f \in A$, then the real and imaginary parts* $\mathrm{Re} f$ *and* $\mathrm{Im} f$ *of $f$ also belong to $A$. Then $A$ is dense in $\mathrm{C}_{\mathbb{R}}(X)$.*

[5]) Let $X = \overline{\mathrm{B}}(0; 1) = \{z \in \mathbb{C} \mid |z \le 1\}$ be the closed unit disc in $\mathbb{C}$ and let $A$ be the set of all functions in $\mathrm{C}_{\mathbb{C}}(X)$ which are analytic in the open unit disc $\mathrm{B}(0; 1)$, i.e. $A := \{f \in \mathrm{C}_{\mathbb{C}}(X) \mid f$ is analytic in $\mathrm{B}(0; 1)\}$. Then $A$ is a closed $\mathbb{C}$-subalgebra of $\mathrm{C}_{\mathbb{C}}(X)$ and $A$ separates points of $X$, but $A \neq \mathrm{C}_{\mathbb{C}}(X)$. (**Hint:** Use *Morera's theorem* to conclude that $A$ is closed in $\mathrm{C}_{\mathbb{C}}(X)$. The complex conjugation belongs $\mathrm{C}_{\mathbb{C}}(X)$, but is not in $A$, since it is not diferentiable at any point.)

**N13.8.** (Arzelà-Ascoli theorem) *Let $F$ be a subset of the space* (*with the uniform metric, see* N13.3) *of $\mathbb{K}$- valued functions on a compact topological space $X$. Suppose that $F$ satisfies the following conditions:*

(1) *For every $x \in X$, the set $\{f(x) \mid f \in F\} \subseteq \mathbb{K}$ is bounded.* (2) *$F$ is equicontinuous.*

*Then every sequence $(f_n)$ with $f_n \in F$ has a uniformly convergent subsequence* (*the limit function* $\lim f_n$ *neend not belong to $F$*), *i.e. $F$ is relatively compact in $\mathrm{C}_{\mathbb{K}}(X)$.*

*Let $F$ be a subset of the space* (*with the uniform metric, see N13.??*) *of $\mathbb{K}$- valued functions on a compact topological space $X$. Then $F$ is compact if and only if the follwoing conditions are fullfilled:*

(1) *For every $x \in X$, the set $\{f(x) \mid f \in F\} \subseteq \mathbb{K}$ is bounded.* (2) *$F$ is equicontinuous.* (3) *$F$ is closed in $\mathrm{C}_{\mathbb{K}}(X)$.*

---

[4]) The compact-open topology was first systematically defined and studied by Fox in 1945 and Arens in 1946.

[5]) This simpleset example requires a little knowledge of the theorey of analytic functions

**13.1.** Let $(X, d)$ be a metric space and let $a \in X$ be a fixed. For each $y \in X$, let $f_y : X \to \mathbb{R}$ be the function defined by $f_y(X) := d(x, y) - d(x, a)$, $x \in X$. Then

**a).** $f_y \in \mathrm{C}_{\mathbb{R}}(X)$ for every $y \in X$ and $\| f_y - f_z \| = d(y, z)$ for all $y, z \in X$.

**b).** The map $\Phi : X \to \mathrm{C}_{\mathbb{R}}(X)$ defined by $y \mapsto f_y$ is an isometry (a distance preservaing map, — the metric on $\mathrm{C}_{\mathbb{R}}(X)$ is the metric of uniform convergence, see Exercise 13.2).

**c).** The closure of $\Phi(X)$ in $\mathrm{C}_{\mathbb{R}}(X)$ is a complete metric space. In particular, $(X, d)$ is isometric to a dense subset of a complete metric space $Y = \overline{\Phi(X)}$. (**Remark:** This give a different proof of the fact that *every metric space has a completion*.)

**13.2.** (Applications of Weierstrass theorem)

**a).** If $f : [0, 1] \to \mathbb{R}$ is a coninuous function and if $\int_0^1 f(x) x^n \, dx = 0$ for all $n = 0, 1, 2, \ldots,$, then show that $f(x) = 0$ for every $x \in [0, 1]$. (**Hint:** It is enough to prove that $\int_0^1 f^2(x) \, dx = 0$. The integral of the product of $f$ with any polynomial is zero. Now, use Weierstrass theorem to conclude that $\int_0^1 f^2(x) \, dx = 0$.)

**b).** The metric space $\mathrm{C}_{\mathbb{R}}([0, 1])$ is separable.

**13.3.** Let $X$ be a topological space and let $F \subseteq \mathrm{C}_{\mathbb{K}}(X)$ be an equicontinuous set of $\mathbb{K}$-valued (continuous) functions on $X$. If for every $x \in X$, the set $\{f(x) \mid f \in F\}$ is bounded, then for every compact subset $K \subseteq X$ the set $\{\| f \|_K \mid f \in F\}$ is bounded.

**13.4.** Prove the following generalisation of Arzela-Ascoli theorem (see N13.8): Let $X$ be a compact topological space and let $Y$ be a complete metric space. A subset $F$ of the space $\mathrm{C}(X, Y)$ of continuous maps from $X$ into $Y$ is compact if and only if the following conditions are fullfilled:
(1) For every $x \in X$, the set $\{f(x) \mid f \in F\} \subseteq Y$ is relatively compact. (2) $F$ is equicontinuous. (3) $F$ is closed in $\mathrm{C}(X, Y)$.

**13.5.** Let $I \subseteq \mathbb{R}$ be a compact interval and let $a \in I$. Further, let $C, L \in \mathbb{R}_+$. The set $F$ of all differentiable functions $f : I \to \mathbb{K}$ with $|f(a)| \leq C$ and $\| f' \|_I \leq L$ is relatively compact in $\mathrm{C}_{\mathbb{K}}(I)$. Is $F$ also closed, i.e. compact?

**13.6.** Let $X$ be a compact metric space. Then the $\mathbb{K}$-Banach-algebra $\mathrm{C}_{\mathbb{K}}(X)$ is separable, i.e. it has a countable dense subset. In particular, the uniform topology $\mathrm{C}_{\mathbb{K}}(X)$ on is second countable. (**Hint:** It is enough to consider the case $\mathbb{K} = \mathbb{R}$. Let $A \subseteq X$ be a countable dense subset of $X$ (why does such a subset exists!). The functions $f_a : x \mapsto d(x, a)$, $a \in A$, separates the points in $X$ and hence generate a dense $\mathbb{R}$- subalgebra of $\mathrm{C}_{\mathbb{R}}(X)$.)

**13.7.** For a topological space $X$, the following statements are equivalent: (1) $X$ is metric and compact. (2) $X$ is compact and there exists a countable family $f_i, i \in I$, of continous functions $f_i : X \to \mathbb{R}$, which separates the points in $X$. (3) $X$ is homeomorphic to a closed subspace of $[0, 1]^{\mathbb{N}}$. (4) $X$ is compact and second countable. (**Hint:** For (1) $\Rightarrow$ (2) see Exercise 13.??. For a proof of (2) $\Rightarrow$ (3) we may assume that $f_i(X) \subseteq [0, 1]$ for all $i \in I$. Then $x \mapsto \big(f_i(x)\big)_{i \in I}$ is an injective continuous map $X \to [0, 1]^I$. The implication (3) $\Rightarrow$ (1) is immediate from (countable) Tychonoff theorem. The implication (4) $\Rightarrow$ (2) follows immediately from the Urysohn's separation lemma. – By passing to the one-point compactfication we get the following important Criterion for metrisability: *A locally compact topological space $X$ with a countable topology is metrisable.* Note that the one-point compcatification $X$ always has a countable topology.)

---

† **Karl Theodor Wilhelm Weierstrass (1815-1897)** was born on Oct 1815 in Ostenfelde, Westphalia (now Germany) and died on 19 Feb 1897 in Berlin, Germany. While at the Gymnasium Weierstrass certainly reached a level of mathematical competence far beyond what would have been expected. He regularly read Crelle's Journal and gave mathematical tuition to one of his brothers. However Weierstrass's father wished him to study finance and so, after graduating from the Gymnasium in 1834, he entered the University of Bonn with a course planned out for him which included the study of law, finance and economics. However, Weierstrass suffered from the conflict of either obeying his father's wishes or studying the subject he loved, namely mathematics. The result of the conflict which went on inside Weierstrass was that he did not attend either the mathematics lectures or the lectures of his planned course. He reacted to the conflict inside him by pretending that he did not care about his studies, and he spent four years of intensive fencing and drinking.

He did study mathematics on his own, however, reading Laplace's Méchanique céleste and then a work by Jacobi on elliptic functions. He came to understand the necessary methods in elliptic function theory by studying transcripts of lectures by Gudermann. In a letter to Lie, written nearly 50 years later, he explained how he came to make the definite decision to study mathematics despite his father's wishes around this time.

*... when I became aware of [a letter from Abel to Legendre] in Crelle's Journal during my student years, [it] was of the utmost importance. The immediate derivation of the form of the representation of the function given by Abel ..., from the differential equation defining this function, was the first mathematical task I set myself; and its fortunate solution made me determined to devote myself wholly to mathematics; I made this decision in my seventh semester ...*

Now Weierstrass had made a decision to become a mathematician but he was still supposed to be on a course studying public finance and administration. After his decision, he spent one further semester at the University of Bonn, his eighth semester ending in 1838, and having failed to study the subjects he was enrolled for he simply left the University without taking the examinations. Weierstrass's father was desperately upset by his son giving up his studies. He was persuaded by a family friend, the president of the law courts at Paderborn, to allow Karl to study at the Theological and Philosophical Academy of Münster so that he could take the necessary examinations to become a secondary school teacher.

On 22 May 1839 Weierstrass enrolled at the Academy in Münster. Gudermann lectured in Münster and was the reason that Weierstrass was so keen to study there. Weierstrass attended Gudermann's lectures on elliptic functions, some of the first lectures on this topic to be given, and Gudermann strongly encouraged Weierstrass in his mathematical studies. Leaving Münster in the autumn of 1839, Weierstrass studied for the teacher's examination which he registered for in March 1840. GUDDERMAN Weierstrass's teacher, in his evaluation wrote: "*With this work the candidate enters the ranks of famous inventors as co-equal*." Gudderman urged publication of the exam project as soon as possible and that would have happened had the philosophy faculty of the royal academy at Münster/Westphalia at that time had the authority to grant degrees. "*Then we would have the pleasure of counting Weierstrass amoung our doctoral graduates*", so was written in 1887 rector's address of Weierstrass' formal pupil W. KILLING (whose name was later immortalized in Lie theory). Not until 1894, fifty-four years after it was written, did Weierstrass publish his exam work.

1842-1848 teacher at the Progymnasium in Deutsch-Krone, West Prussia, of mathematics, penmanship and gymnastics; 1848-1855 teacher at the Gymnasium in Braunsberg, East Prissia;1854 publication of trail-blazing results (gotten already in 1849) "Zur Theorie der Abelschen Functionen" in Journal für Reine und Angew. Math., thereupon honorary doctorate from the University of Königberg and promotion to assistant headmaster; 1856 at the instigation of A. VON HUMBOLT and L. CRELE appointment as professor at the Industrial Institute (later Techinal University) at Berlin; 1857 adjunct professor at the University of Berlin; after 1860 lectures often with more than 200 auditors; 1861 breakdown from over work; 1864 at the age of almost fifty appointmnet to an ordinary professorship, created for him, at the University of Berlin; 1873/74 rector magnificus there, member of numerous academies at home and abroad;1885 stamping of Weierstrass medal (for his 70th birthday); 1890 teaching activity halted by serious illness, confinement to wheelchair; 1895 festive unveilling his image in the national gallery (80th birthday); 1897 died in Berlin.

Unfortunately, unlike CAUCHY, Weierstrass never wrote his lectures out in the book form , but there are transcriptions by his various puplis, for example, from H. A. SCHWARZ's hand there is an elaboration of his lectures on *Differentialrechnung* held at the Royal Industrial Institute in the 1861 summer semester. There is also a transcription by A. HURWITZ of his summer semester 1878 lectures on *Einleitung in die theorie der analytischen Funktionen* and another by W. KILLING. Weierstrass's lectures became world famous; when in 1873 –two years after the Franko-Prussian War –MITTAG-LEFFLER came to Paris to study, HERMITE told him: "*you have made a mistake, sir; you should have attended Weierstrass' course in Berlin. He is the master of us all*".

There is no exhaustive biography of Weierstrass, but in the personal remarks made by A. KNESER describes the mathematical life in the 1880's thus: *The undisputed master of the whole opeartion was without doubt Weierstrass, a regal and in every way imposing figure. All knew the magnificent white-locked head, the shining blue eyes slightly dropping at the corners which belonged to the country boy of pure Westphalian stock. By this time his lectures had evolved to a high level of perfection in presentation as well as content and only seldom were those tense minutes experienced where the great man flatered and even the promptings of his faithful assistant at the balckboard, perhaps my friend Richard Müller, couldn't get him back on the track; then he would sink into majestic silence for few minutes; two hundred pairs of young eyes were riveted on the splendid brow with the devout conviction that behind that shining facade the greatest intellect was at work. There were in fact, two hundred youths who attended and listened intently to Weierstrass's lectures on elliptic functions, fully aware that at that time such things never came up on any state examination, a dazzling testimonial to intellectual sprit of times. People even knew very little about the applications of these things, although there were already available some very beautiful ones. The doctrine of the primacy of applied mathematics, of the greater worth of applications as against pure mathematics, had not yet been discovered. The humor of the young was unleashed even on this great man: he was considered a connoisseur of wine and the Berliners, who mocked his westphalian pronunciation, claimed to have actually heard from him the following quintessential example: I'd kladly kulp a kood klass of Burkundy.* – the k's here should be read as g's.

Weierstrass by his lectures in Berlin, influenced mathematics in Germany like no one else. The assistant headmaster from East prussia became the "praeceptor mathematicus Germaniae".

---

†† **Marshall Harvey Stone** was born on 8 April 1903 in New York, USA and died on 9 Jan 1989 in Madras, India Marshall Stone's father was a distinguished lawyer and the family tradition would have had him follow his father's subject. He studied at Harvard from 1919 to 1922, then was appointed an instructor at Harvard for session 1922/23 to see whether he would enjoy teaching mathematics and whether he would take his mathematical studies further.

Indeed he did rapidly decide that he wanted to pursue a career in mathematics and studied for his doctorate under Birkhoff. His doctorate was awarded in 1926 for a thesis entitled Ordinary Linear Homogeneous Differential Equations of Order n and the Related Expansion Problems. By 1925 he was appointed to Columbia University, in 1927 to Harvard. During this period Stone's interests followed very much those of his research supervisor Birkhoff. He published eleven papers on the theory of orthogonal expansions between 1925 and 1928. In these papers a special role is played by expansions in terms of the eigenfunctions of linear differential operators.

Although he would return to Harvard again in 1933, Stone first accepted a post at Yale from 1931 to 1933. Back at Harvard in 1933 he was promoted to full professor there in 1937.

During these years Stone's research took a number of directions. From 1929 he worked on self-adjoint operators in Hilbert space and included his results in a major publication of a 600 page book Linear transformations in Hilbert space and their applications to analysis.

In 1932 he proved results on spectral theory, arising from group theoretical methods in quantum mechanics, which had been conjectured by Weyl. Then in 1934 he published two papers on Boolean algebras. He made this study while attempting to understand more deeply the basics underlying his results on spectral theory.

One particularly important result proved by Stone during this period was a substantial generalisation of Weierstrass's results on uniform approximation of continuous functions by polynomials. This result is now known as the Stone-Weierstrass theorem.

During World War II Stone undertook secret war work and then in 1946 he left Harvard to take up the chairmanship of the mathematics department at the University of Chicago. He did an outstanding job in returning this famous research school to the eminence it had attained earlier by making appointments such as Weil, Chern and Mac Lane.

From 1952 Stone stepped down as head of department in favour of Mac Lane but he remained at Chicago until he retired in 1968. His interests, which included cooking, are described: *Of all Stone's many interests his love of travel was surely dominant. He began to travel when he was quite young and was on a trip to India when he died. ... Marshall Stone was a man with a very broad outlook and a wide range of interests who seems to have thought rather deeply about a number of issues. ... here was an unusually thoughtful man with a high degree of penetration and insight. ... he seemed well endowed with a quality which I can only describe as wisdom.*