MA-231 Topology

14. Separation Axioms[1])

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First recall the following definitions and results:

N14.1. (Regular spaces ; T_3-spaces[2])) Let X be a topological space. We say that X is regular space if whenever A is a closed subset in X and $x \notin A$, then there are disjoint open subsets U and V with $x \in U$ and $A \subseteq V$. A regular T_1 space is called a T_3-space. It is clear that every T_3-space is T_2. But not every T_2-space is T_3, for example,

a). For a topological space the following statements are equivalent: (i) X is regular. (ii) If U is open in X and if $x \in U$, then there is an open subset V containing x such that $\overline{V} \subseteq U$. (iii) Each $x \in X$ has a nhood base consisting of closed sets.

b). (1) Every subspace of a regular (resp. T_3) space is completely regular (resp. T_3). (2) A non-empty product space is regular (resp. T_3) space if and only if each factor space is regular (resp. T_3). (3) Quotients of T_3 spaces need not be regular.

N14.2. (Completely regular spaces ; $\mathrm{T}_{3\frac{1}{2}}$ – Tychonoff spaces[3])) Let X be a topological space. We say that X is completely regular space if whenever A is a closed subset in X and $x \notin A$, then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$. It is clearly enough to find a continuous function $F : X \to \mathbb{R}$ such that $f(x) = b$ and $f(A) = a$, where $b \neq a$; any such function will be said to separate A and x. A completely regular T_1 space is called a Tychonoff space[4]).

a). Completely regular spaces are regular. (**Remark:** Not every regular space is completely regular. There is more complicated example of a T_3 space on which every continuous real-valued function is constant!)

b). *Every metric space (X, d) is Tychnoff.* In fact, if A is a closed subset of X and $x \notin A$, then $f = d(-, A) : X \to \mathbb{R}$ is a continuous function such that $f(A) = 0$ and $f(x) \neq 0$.

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[1]) The T_0 axiom is usually credited to KOLMOGOROFF and the T_1 axiom to FRECHET or RIESZ (and spaces satisfying these axioms are sometimes called *Kolmogoroff spaces*, *Frechet spaces* or *Riesz spaces*. TIEZTE was the first use the term "separation axiom" (or, in German *Trennungsaxiom*) in 1923. The terminology using increasing subscripts for incresingly stronger conditions is due to ALEXANDROFF and HOPF. The T_2 axiom, introduced by HAUSDORFF under the name "separability" is probably the most useful separation axiom especially where convergence is concerned. HAUSDORFF even included this axiom in the original axioms for a topology given by him. Indeed many topologists do not bother to consider spaces which are not Hausdorff. This attitude is typified in the BOURBAKI school where "regular" means what we call T_3, "normal" means what we call T_4 and even the definition of compactness includes Hausdorff property! In 1944 DIEUDONNE defined "paracompactness" which is a much stronger condition than normality.

[2]) Regular spaces were first introduced by VIETORIS in 1921.

[3]) Completely regular spaces were first considered by URYSOHN in 1925. Their importance was established with the proof of the celebrated *embedding theorem* of TYCHONOFF (see N14.2 -e)) in 1929. The name "Tychonoff" was suggested by TUKEY.

[4]) An early joke has somehow become semistandard, with some writers referring to Tychonoff space as $\mathrm{T}_{3\frac{1}{2}}$-spaces.

c). (1) Every subspace of a completely regular (resp. Tychonoff) space is completely regular (resp. Tychonoff). (2) A non-empty product space is completely regular (resp. Tychonoff) space if and only if each factor space is completely regular (resp. Tychonoff). (3) Quotients of Tychonoff spaces need not be completely regular or T_2.

d). A topological space X is completely regular if and only if it has the weak topology induced by its family $C_{bdd}(X)$ of bounded real-valued continuous functions.

e). (Tychonoff's embedding theorem[5])) *A topological space X is Tychonoff if and only if it is homeomorphic to some subspace of some cube.*

N14.3. (Normal spaces ; T_4-spaces[6])) Let X be a topological space. We say that X is normal if whenever A and B are disjoint closed subsets in X, there are disjoint opne subsets U and V in X such that $A \subseteq U$ and $B \subseteq V$. A normal T_1 space is called a T_4-space.

a). *Every metric space* (X, d) *is* T_4. (**Proof:** If A and B are disjoint closed subsets in X. For each $x \in A$, chosse $\delta_x > 0$ such that $B(x; \delta_x) \cap B = \emptyset$ and for each $y \in B$, chosse $\varepsilon_y > 0$ such that $B(y; \varepsilon_y) \cap A = \emptyset$. Then $U = \cup_{x\in A} B(x, \frac{\delta_x}{3})$ and $V = \cup_{y\in B} B(y, \frac{\varepsilon_y}{3})$ are open sets in X conatining A and B respectively. Suppose $z \in U \cap V$. Then $d(x, z) < \frac{\delta_x}{3}$ and $d(y, z) < \frac{\varepsilon_y}{3}$ and so $d(x, y) < \frac{\delta_x}{3} + \frac{\varepsilon_y}{3} < \delta_x$, assuming that $\delta_x = \max\{\delta_x, \varepsilon_y\}$. But then $y \in B(x; \delta_x)$ a contradiction. Therefore $U \cap V = \emptyset$. This proves that X is normal.)

b). (1) Arbitrary subspaces of a normal (resp. T_4) spaces need not be normal (resp. T_4). But every closed subspace of a normal (resp. T_4) space is normal (resp. T_4). If every subspace of a topological space is normal, then X is said to be completely normal[7]). (2) Products of normal spaces need not be normal. (3) Arbitrary quotients of T_4 spaces need not be T_4. (3) The closed continuous image of a normal (resp. T_4) space is normal (resp. T_4).

N14.4. (Urysohn's characterisation of Normality --- Urysohn's lemma[8]) *Let X be a topological space. Then X is normal if and only if whenever A and B are closed subsets in X, there is a continuous function* $f : X \to [0, 1]$ *such that* $f(A) = 0$ *and* $f(B) = 1$. *In particular, every* T_4 *space is Tychonoff.* (**Remark:** Such a function is called a Urysohn function. Note that it is nowhere claimed, nor, is it true, that f is 0 only on A and 1 only on B. It other words we merely claim that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$. There may be points outside $A \cup B$ at which f is either 0 or 1. See also N14.6.)

N14.5. (Tietze's characterisation of Normality --- Tietze's extension theorem[9]) *Let X be a topological space. Then X is normal if and only if whenever A is a closed subset in X and* $f : A \to \mathbb{R}$ *is a continuous function, there is a continuous extension of f to all X, i.e. there is a continuous function* $F : X \to \mathbb{R}$ *such that* $F|A = f$.

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14.1. (Examples on Regularity and Complete regularity) (1) The slotted plane is T_2 but not T_3 . (2) The looped line is Tychonoff. (3) The Moore plane is Tychonoff.

14.2. (Zero-sets in completely regular spaces) Let X be a topological space. A zero-set in a topological space X is a set of the form $f^{-1}(0)$ for some continuous function $f : X \to \mathbb{R}$.

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[5]) This is the celebrated embedding theorem of Tychonoff was indeed proved by TYCHONOFF in 1929.
[6]) The T_4-spaces were introduced by TIEZTE in 1923.
[7]) Complete normality was added to the list of separation axioms in 1923 by TIETZE
[8]) This is a non-trivial result is due to URYSOHN; its analogue does not hold for regular spaces. In fact there do exist regular (even T_3- spaces) on which every continuous real valued function is constant (see for example N14.2 -a)). Another novel feature of this theorem guarantees the existence of a real valued function with certain properties. The desired function is to be constructed from subsets of the normal space and the construction is nothing short of ingenious. Despite of its non- triviality, it is called Urysohn's lemma because Urysohn used it as a lemma to prove **Urysohn's metrisation theorem :** *A second countable topological space is metrisable if and only if it is* T_3. — We urge the reader to read the proof of this theorem again and again until he can do justice to this beautiful theorem of URYSOHN who incidently, died at the age of twenty six!.
[9]) Tietze extension theorem is a classic theorem in topology and analysis and was proved by TIETZE in 1915. This theorem has been extended in several ways.

a). If $f : X \to \mathbb{R}$ is a real-valued continuous functionm then the sets $\{x \in X \mid f(x) \geq a\}$ and $\{x \in X \mid f(x) \leq a\}$ are zero-sets for each $a \in \mathbb{R}$. (**Hint:** $g(x) := \max\{f(x) - a, 0\}$ is a continuous function.)

b). X is a completely regular space if and only if each point $x \in X$ has a nhood base consisting of complements of zero-sets.

c). X is a completely regular space if and only if the zero-sets form a base for closed subsets in X, i.e. every closed subset in X is an intersection of zero-sets.

14.3. (Completely Hausdorff spaces) A topological space X is called completely Hausdorff or functionally Hausdorff if whenever $x, y \in X$, $x \neq y$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. Every completely Hausdorff space is Hausdorff. (**Remark:** the famous example of E. HEWITT of a regular T_1-space in which every continuous real-valued function is constant (see N14.2 -a)) shows that not every regular T_1-space is completely Hausdorff.)

14.4. (Examples on Normality) **a).** A normal space need not be regular. (**Hint:** Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\}$. Then X is normal, since no two non-empty closed subsets are disjoint. But X is not regular, since the point 1 cannot be separted from the closed subset $(-\infty, 0]$ by disjoint subsets.)

b). Let X be the Sorgenfrey line. Then $X \times X$ is not normal.

c). Give an example of a topological space X, a closed subset A in X and a continuous real-valued function $f : A \to \mathbb{R}$ which cannot be extened continuously to X. (**Hint:** Let $X = [0, 1]$, $A = (0, 1]$ and $f : A \to \mathbb{R}$, $x \mapsto \sin(\frac{1}{x})$. This f cannot be extended continuously to X, since there exist sequences $x_n, n \in \mathbb{N}$ and $y_n, n \in \mathbb{N}$ in A which converge to 0 in X such that the sequences $f(x_n), n \in \mathbb{N}$ and $f(y_n), n \in \mathbb{N}$ converge to distinct limits in $\mathbb{R}$.)

14.5. (Completely normal spaces) A topological space is called completely normal if every subspace of X is normal.

a). A topological space is completely normal if and only if whenever A and B are subsets in X with $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, then there are disjoint open subsets $U \supseteq A$ and $V \supseteq B$. (**Hint:** To do necessity, consider the subspace $X \setminus (\overline{A} \cap \overline{B})$ which contains both A and B and in which A and B have disjoint closures. Sufficiency is easy.)

b). Every metric space is completely normal.

14.6. (Perfectly normal spaces[10])) We say that a T_1-space X is perfectly normal if whenever A and B are disjoint closed subsets in X, there is a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

a). A topological space X is perfectly normal if and only if it is T_4 and each closed set in X is a G_δ-set, i.e. a countable intersection of open sets. (**Hint:** For sufficiency, if A is a closed set and $A = \cap_{n \in \mathbb{N}} G_n$, where each G_n is open, then for each $n \in \mathbb{N}$, there exists a Urysohn function f_n such that $f_n(A) = 0$ and $f_n(X \setminus G_n) = 1$. Set $f_A(x) = \sum_{n \in \mathbb{N}} \frac{f_n(x)}{2^n}$. Now, if A and B are disjoint closed sets, then the function f defined by $f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$ is a continuous function on X such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.)

b). Every metric space is perfectly normal.

14.7. Every ordered space is T_4.

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† **Frigyes Riesz (1880-1956)** was born on 22 Jan 1880 in Győr, Austria-Hungary (now Hungary) and died on 28 Feb 1956 in Budapest, Hungary. Frigyes Riesz's father Ignácz Riesz was a medical man and Frigyes's his younger brother, Marcel Riesz, was himself a famous mathematician. Frigyes (or Frederic in German) Riesz studied at Budapest. He went to Göttingen and Zurich to further his studies and obtained his doctorate from Budapest in 1902. His doctoral dissertation was on geometry. He spent two years teaching in schools before being appointed to a university post. Riesz was a founder of functional analysis and his work has many important applications in physics. He built on ideas introduced by Fréchet in his dissertation, using Fréchet's ideas of distance to provide a link between Lebesgue's work on real functions and the area of integral equations developed by Hilbert and his student Schmidt.

In 1907 and 1909 Riesz produced representation theorems for functional on quadratic Lebesgue integrable functions and, in the second paper, in terms of a Stieltjes integral. The following year he introduced the space of q-fold Lebesgue integrable functions and so he began the study of normed function spaces, since, for $q \geq 3$ such spaces are not Hilbert spaces. Riesz

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[10]) It is not in gerneral true that: given disjoint closed subsets A and B in a normal space, there will be a Urysohn function such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Perfect normality was introduced by ALEXANDROOF and URYSOHN. URYSOHN proved every perfectly normal space is completely normal.

introduced the idea of the 'weak convergence' of a sequence of functions $(f_n(x))$. A satisfactory theory of series of orthonormal functions only became possible after the invention of the Lebesgue integral and this theory was largely the work of Riesz.

Riesz's work of 1910 marks the start of operator theory. In 1918 his work came close to an axiomatic theory for Banach spaces, which were set up axiomatically two years later by Banach in his dissertation. Riesz was appointed to a chair in Kolozsvár in Hungary in 1911. However, the Hungarian government was forced to sign the Treaty of Trianon on 4 June, 1920. Hungary was left with less than one third of the land that had previously been Hungary. Romania, Czechoslovakia and Yugoslavia all took over large areas but Austria, Poland and Italy also gained land from Hungary. Kolozsvár was no longer in Hungary after the Treaty of Trianon but rather it was in Romania and was renamed Cluj, so the Hungarian University there had to move within the new Hungarian borders and it moved to Szeged in 1920, where there had previously been no university.

In Szeged in 1922 Riesz set up the János Bolyai Mathematical Institute in a joint venture with Haar. Of course the Institute was named after the famous Hungarian mathematician whose birthplace was Kolozsvár, the town from which the university had just been forced to move. Riesz became editor of the newly founded journal of the Institute Acta Scientiarum Mathematicarum which quickly became a major source of mathematics. Riesz was to publish many papers in this journal, the first in 1922 being on Egorov's theorem on linear functionals. It was published in the first part of the first volume.

In 1945 Riesz was appointed to the chair of mathematics in the University of Budapest. Many of Riesz's fundamental findings in functional analysis were incorporated with those of Banach. His theorem, now called the Riesz- Fischer theorem, which he proved in 1907, is fundamental in the Fourier analysis of Hilbert space. It was the mathematical basis for proving that matrix mechanics and wave mechanics were equivalent. This is of fundamental importance in early quantum theory.

Riesz made many contributions to other areas including ergodic theory where he gave an elementary proof of the mean ergodic theorem in 1938. He also studied orthonormal series and topology. Rogosinski writes of Riesz's style: *The work of F Riesz is not only distinguished by the genuine importance of his results, but also by his aesthetic discernment in mathematical taste and diction. ... The more leisurely mastership of F Riesz's style, whether he writes in his native Hungarian, or in French or German, conveys such pleasure and is to the older mathematician a nostalgic remainder of what we are in danger to lose. For him there was no mere abstraction for the sake of a structure theory, and he was always turning back to the applications in some concrete and substantial situation.*

His book Leçon's d'analyse fonctionnelle is one of the most readable accounts of functional analysis ever written. Rogosinski describes this book which Riesz wrote jointly with his student B Szökefalvi-Nagy as follows: *Here, in the first half written by himself, we find the old master picturing to us Real Analysis as he saw it, lovingly, leisurely, and with the discerning eye of an artist. This book, I have no doubt, will remain a classic in the treasure house of mathematical literature. With it, and with all his other work, will live the memory of Frederic Riesz as a great and fertile mathematician for long in the history of our art.*

Riesz received many honours for his work. He was elected to the Hungarian Academy of Sciences and, in 1949, he was awarded its Kossuth Prize. He was elected to the Paris Academy of Sciences and to the Royal Physiographic Society of Lund in Sweden. He received honorary doctorates from the universities of Szeged, Budapest and Paris.

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†† **Andrey Nikolaevich Kolmogorov (1903-1987)** was born on 25 April 1903 in Tambov, Tambov province, Russia and died on 20 Oct 1987 in Moscow, Russia. Andrei Nikolaevich Kolmogorov's parents were not married and his father took no part in his upbringing. His father Nikolai Kataev, the son of a priest, was an agriculturist who was exiled. He returned after the Revolution to head a Department in the Agricultural Ministry but died in fighting in 1919. Kolmogorov's mother also, tragically, took no part in his upbringing since she died in childbirth at Kolmogorov's birth. His mother's sister, Vera Yakovlena, brought Kolmogorov up and he always had the deepest affection for her.

In fact it was chance that had Kolmogorov born in Tambov since the family had no connections with that place. Kolmogorov's mother had been on a journey from the Crimea back to her home in Tunoshna near Yaroslavl and it was in the home of his maternal grandfather in Tunoshna that Kolmogorov spent his youth. Kolmogorov's name came from his grandfather, Yakov Stepanovich Kolmogorov, and not from his own father. Yakov Stepanovich was from the nobility, a difficult status to have in Russia at this time, and there is certainly stories told that an illegal printing press was operated from his house.

After Kolmogorov left school he worked for a while as a conductor on the railway. In his spare time he wrote a treatise on Newton's laws of mechanics. Then, in 1920, Kolmogorov entered Moscow State University but at this stage he was far from committed to mathematics. He studied a number of subjects, for example in addition to mathematics he studied metallurgy and Russian history. Nor should it be thought that Russian history was merely a topic to fill out his course, indeed he wrote a serious scientific thesis on the owning of property in Novgorod in the 15th and 16th centuries. There is an anecdote told by D G Kendall regarding this thesis, his teacher saying: *You have supplied one proof of your thesis, and in the mathematics that you study this would perhaps suffice, but we historians prefer to have at least ten proofs.*

Kolmogorov may have told this story as a joke but nevertheless jokes are only funny if there is some truth in them and undoubtedly this is the case here. In mathematics Kolmogorov was influenced at an early stage by a number of outstanding mathematicians. P S Aleksandrov was beginning his research (for the second time) at Moscow around the time Kolmogorov began his undergraduate career. Luzin and Egorov were running their impressive research group at this time which the students called 'Luzitania'. It included M Ya Suslin and P S Urysohn, in addition to Aleksandrov. However the person who made the deepest impression on Kolmogorov at this time was Stepanov who lectured to him on trigonometric series.

It is remarkable that Kolmogorov, although only an undergraduate, began research and produced results of international importance at this stage. He had finished writing a paper on operations on sets by the spring of 1922 which was a major generalisation of results obtained by Suslin. By June of 1922 he had constructed a summable function which diverged almost everywhere. This was wholly unexpected by the experts and Kolmogorov's name began to be known around the world. Some authors note that: *Almost simultaneously [Kolmogorov] exhibited his interest in a number of other areas of classical analysis: in problems of differentiation and integration, in measures of sets etc. In every one of his papers, dealing with such a variety of topics, he introduced an element of originality, a breadth of approach, and a depth of thought.*

Kolmogorov graduated from Moscow State University in 1925 and began research under Luzin's supervision in that year. It is remarkable that Kolmogorov published eight papers in 1925, all written while he was still an undergraduate. Another milestone occurred in 1925, namely Kolmogorov's first paper on probability appeared. This was published jointly with Khinchin and contains the 'three series' theorem as well as results on inequalities of partial sums of random variables which would become the basis for martingale inequalities and the stochastic calculus.

In 1929 Kolmogorov completed his doctorate. By this time he had 18 publications and Kendall writes: *These included his versions of the strong law of large numbers and the law of the iterated logarithm, some generalisations of the operations of differentiation and integration, and a contribution to intuitional logic. His papers ... on this last topic are regarded with awe by specialists in the field. The Russian language edition of Kolmogorov's collected works contains a retrospective commentary on these papers which [Kolmogorov] evidently regarded as marking an important development in his philosophical outlook.*

An important event for Kolmogorov was his friendship with Aleksandrov which began in the summer of 1929 when they spent three weeks together. On a trip starting from Yaroslavl, they went by boat down the Volga then across the Caucasus mountains to Lake Sevan in Armenia. There Aleksandrov worked on the topology book which he co-authored with Hopf, while Kolmogorov worked on Markov processes with continuous states and continuous time. Kolmogorov's results from his work by the Lake were published in 1931 and mark the beginning of diffusion theory. In the summer of 1931 Kolmogorov and Aleksandrov made another long trip. They visited Berlin, Göttingen, Munich, and Paris where Kolmogorov spent many hours in deep discussions with Paul Lévy. After this they spent a month at the seaside with Fréchet

Kolmogorov was appointed a professor at Moscow University in 1931. His monograph on probability theory Grundbegriffe der Wahrscheinlichkeitsrechnung published in 1933 built up probability theory in a rigorous way from fundamental axioms in a way comparable with Euclid's treatment of geometry. One success of this approach is that it provides a rigorous definition of conditional expectation. As noted: *The year 1931 can be regarded as the beginning of the second creative stage in Kolmogorov's life. Broad general concepts advanced by him in various branched of mathematics are characteristic of this stage.*

After mentioning the highly significant paper Analytic methods in probability theory which Kolmogorov published in 1938 laying the foundations of the theory of Markov random processes, they continue to describe: *... his ideas in set-theoretic topology, approximation theory, the theory of turbulent flow, functional analysis, the foundations of geometry, and the history and methodology of mathematics. [His contributions to] each of these branches ... [is] a single whole, where a serious advance in one field leads to a substantial enrichment of the others.*

Aleksandrov and Kolmogorov bought a house in Komarovka, a small village outside Moscow, in 1935. Many famous mathematicians visited Komarovka: Hadamard, Fréchet, Banach, Hopf, Kuratowski, and others. Gnedenko and other graduate students went on : *... Mathematical outings [which] ended in Komarovka, where Kolmogorov and Aleksandrov treated the whole company to dinner. Tired and full of mathematical ideas, happy from the consciousness that we had found out something which one cannot find in books, we would return in the evening to Moscow.*

Around this time Malcev and Gelfand and others were graduate students of Kolmogorov along with Gnedenko who describes what it was like being supervised by Kolmogorov: *The time of their graduate studies remains for all of Kolmogorov's students an unforgettable period in their lives, full of high scientific and cultural strivings, outbursts of scientific progress and a dedication of all one's powers to the solutions of the problems of science. It is impossible to forget the wonderful walks on Sundays to which [Kolmogorov] invited all his own students (graduates and undergraduates), as well as the students of other supervisors. These outings in the environs of Bolshevo, Klyazma, and other places about 30-35 kilometres away, were full of discussions about the current problems of mathematics (and its applications), as well as discussions about the questions of the progress of culture, especially painting, architecture and literature.*

In 1938-1939 a number of leading mathematicians from the Moscow University joined the Steklov Mathematical Institute of the USSR Academy of Sciences while retaining their positions at the University. Among them were Aleksandrov, Gelfand, Kolmogorov, Petrovsky, and Khinchin. The Department of Probability and Statistics was set up at the Institute and Kolmogorov was appointed as Head of Department.

Kolmogorov later extended his work to study the motion of the planets and the turbulent flow of air from a jet engine. In 1941 he published two papers on turbulence which are of fundamental importance. In 1954 he developed his work on dynamical systems in relation to planetary motion. He thus demonstrated the vital role of probability theory in physics.

We must mention just a few of the numerous other major contributions which Kolmogorov made in a whole range of different areas of mathematics. In topology Kolmogorov introduced the notion of cohomology groups at much the same time, and independently of, Alexander. In 1934 Kolmogorov investigated chains, cochains, homology and cohomology of a finite cell complex. In further papers, published in 1936, Kolmogorov defined cohomology groups for an arbitrary locally compact topological space. Another contribution of the highest significance in this area was his definition of the cohomology ring which he announced at the International Topology Conference in Moscow in 1935. At this conference both Kolmogorov and Alexander lectured on their independent work on cohomology.

In 1953 and 1954 two papers by Kolmogorov, each of four pages in length, appeared. These are on the theory of dynamical systems with applications to Hamiltonian dynamics. These papers mark the beginning of KAM-theory, which is named after Kolmogorov, Arnold and Moser. Kolmogorov addressed the International Congress of Mathematicians in Amsterdam in 1954 on this topic with his important talk General theory of dynamical systems and classical mechanics.

N H Bingham notes Kolmogorov's major part in setting up the theory to answer the probability part of Hilbert's Sixth Problem "to treat ... by means of axioms those physical sciences in which mathematics plays an important part; in the first rank are the theory of probability and mechanics" in his 1933 monograph Grundbegriffe der Wahrscheinlichkeitsrechnung. Bingham also notes: *... Paul Lévy writes poignantly of his realisation, immediately on seeing the "Grundbegriffe", of the opportunity which he himself had neglected to take. A rather different perspective is supplied by the eloquent writings of Mark Kac on the struggles that Polish mathematicians of the calibre Steinhaus and himself had in the 1930s, even armed with the "Grundbegriffe", to understand the (apparently perspicuous) notion of stochastic independence.*

If Kolmogorov made a major contribution to Hilbert's sixth problem, he completely solved Hilbert's Thirteenth Problem in 1957 when he showed that Hilbert was wrong in asking for a proof that there exist continuous functions of three variables which could not be represented by continuous functions of two variables.

Kolmogorov took a special interest in a project to provide special education for gifted children: *To this school he devoted a major proportion of his time over many years, planning syllabuses, writing textbooks, spending a large number of teaching hours with the children themselves, introducing them to literature and music, joining in their recreations and taking them on hikes, excursions, and expeditions. ... [Kolmogorov] sought to ensure for these children a broad and natural development of the personality, and it did not worry him if the children in his school did not become mathematicians. Whatever profession they ultimately followed, he would be content if their outlook remained broad and their curiosity unstifled. Indeed it must have been wonderful to belong to this extended family of [Kolmogorov].*

Such an outstanding scientist as Kolmogorov naturally received a whole host of honours from many different countries. In 1939 he was elected to the USSR Academy of Sciences. He received one of the first State Prizes to be awarded in 1941, the Lenin Prize in 1965, the Order of Lenin on six separate occasions, and the Lobachevsky Prize in 1987. He was also elected to the many other academies and societies including the Romanian Academy of Sciences (1956), the Royal Statistical Society of London (1956), the Leopoldina Academy of Germany (1959), the American Academy of Arts and Sciences (1959), the London Mathematical Society (1959), the American Philosophical Society (1961), The Indian Statistical Institute (1962), the Netherlands Academy of Sciences (1963), the Royal Society of London (1964), the National Academy of the United States (1967), the French Academy of Sciences (1968).

In addition to the prizes mentioned above, Kolmogorov was awarded the Balzan International Prize in 1962. Many universities awarded him an honorary degree including Paris, Stockholm, and Warsaw. Kolmogorov had many interests outside mathematics, in particular he was interested in the form and structure of the poetry of the Russian author Pushkin.