A Peep into Four Dimensional Space

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Dimension of a Space Through Examples

1. Zero dimensional space: A point.
2. One dimensional space: A straight line. A train moves in one dimensional space - forward or backward.
3. Two dimensional space: A plane. A ship on the surface of ocean moves in two dimension space.
4. Three dimensional space: An aircraft flies in 3-dimensional space.
5. Where is an example of a four dimensional space? A mathematician peeps into it with his imagination and that is what we shall do in this lecture.

Two ways to look at the dimension of a space

- There are two ways to talk of dimension of a space:
- 1. algebraically - mathematicians find it very simple this way,
- 2. geometrically, which non-mathematicians accept easily but only up to $n = 3$.
- We cannot comprehend n-dimensional space geometrically, when $n \geqq 4$. Because, there is no physical space of 4 or more dimensions.

4-Dimensional Space of relativity

- A great revolution took place in 1905 with discovery of the theory of relativity by Albert Einstein. Before that, time and space were assumed to be independent.
- Einstein explained that for persons in relative motion: there is no absolute time, and space and time get mingled up. Physicists use phrase "four dimensional space: (x, y, z, t)" of relativity or simply space-time. It is a mathematical idea.
- Due to this people, who do not understand SRT, think that 4- dimensional space exists geometrically, only they are unable to comprehend it.

4-Dimensional Space of relativity ··· cont.

- I would like to make it clear that our topic "four dimensional space" is purely mathematical and has no physical reality.
- The *space-time* of the theory of relativity only means that the real 3-D space gets mingled with time.
- For a mathematician, it is a simple matter to define these concepts.
- Before that we need to see the relation between real line and the set of real numbers.

Popular idea of n-dimensional space

- 0-dimensional space: a point.
- 1-dimensional space: a line.
- 2-dimensional space: a plane.
- 3-dimensional space: space where we live.
- 4-dimensional space: we do not know.

Real number system and a line

- Notation: $\mathbb{R}$ = set of real numbers = set of rationals + set of irrationals.
- Notation: $\mathbb{R}^n = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{R},\ i = 1, 2, \cdots, n$.
- $(., ., .)$ is symbol in which the entries have definite order. It is a vector with n components.
- There is one to one correspondence between points on a line and $\mathbb{R}$.
- Therefore we denote one dimensional space, i.e., a line by $\mathbb{R}$. Note this convenient notation.

Coordinates in 3-dimensional space

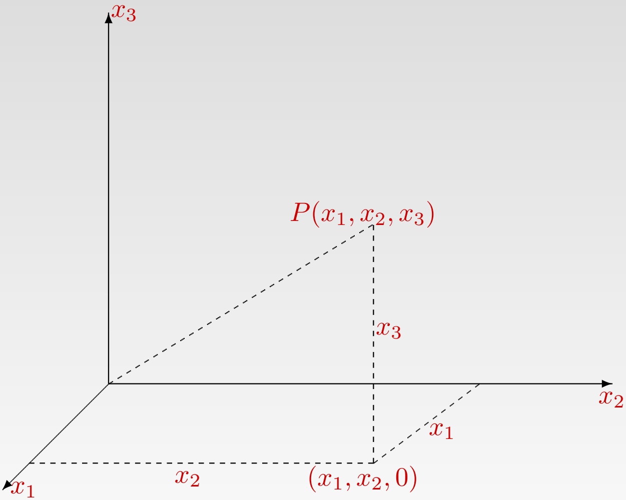
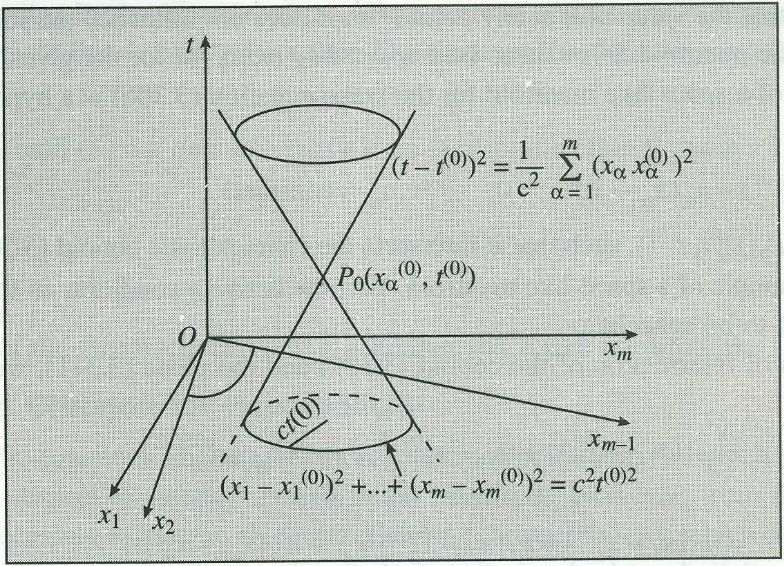


Figure: One to one correspondence between $\mathbb{R}^3$ and 3-D space. Geometrical visualization of components of coordinates for $\mathbb{R}^n$ is possible as shown above.

Visualisation of an Object in n-D Space



Mathematician's definition of n-dimensional space

- Thus 1-dimensional space is denoted by $\mathbb{R}$.
- A point in a plane is represented by an ordered pair of real numbers, i.e., 2-dimensional space is denoted by $\mathbb{R}^2$.
- A points in space is represented by an ordered triple of real numbers. Thus 3-dimensional space is denoted by $\mathbb{R}^3$.
- $\mathbb{R}^4$ is a 4-dimensional space. Note the change in sequence of words.
- . . . .
- $\mathbb{R}^n$ is a n-dimensional space.

Mathematician's definition of n-dimensional space

Mathematicians definition of n-dimensional space is simple and elegant, and without any ambiguity.

An example from mathematics

- Consider the set of all polynomials of degree 2 or less with real coefficients: $a_0 + a_1 x + a_2 x^2$?
- In order to get a deeper understanding of the structure of this set, a mathematician formulates this as:
- Consider the space of all polynomials of degree 2 or less with real coefficients.
- Then he asks: What is the dimension of he space of all polynomials of degree 2 or less with real coefficients.
- Is it a meaningful or meaningless quetion? Can we call any set as "space" and ask for its dimension?

An example from mathematics

- Consider the space of all polynomials of degree 2 or less with real coefficients: $a_0 + a_1 x + a_2 x^2$.
- Given a polynomial $a_0 + a_1 x + a_2 x^2$, we get a triplet (a_0, a_1, a_2).
- Given a triplet (a_0, a_1, a_2) we can construct a polynomial $a_0 + a_1 x + a_2 x^2$.
- The space of polynomials of degree 2 or less is in one to one correspondence with the the space of ordered triplets.

Polynomials of degree 2 or less with real coefficients

- When $a_1 = 0, a_2 = 0$, we have a plynomial of degree 0, which corresponds to a point $(a_0, 0, 0)$ on the x_1-axis.
- When $a_2 = 0$, we have a polynomial of degree 1 or less, which corresponds to a point $(a_0, a_1, 0)$ in the (x_1, x_2)-plane.
- A polynomial of degree 2, $a_2 \neq 0$, corresponds to a general point (a_1, a_2, a_3) in 3-D .

- The space of of all polynomials with real coefficients of degree 3 or less is 3-dimensional.
- Dimension of plynomials of degree zero, i.e. a_0, and that of polynomials of degree 1 or less, i.e. $a_0 + a_1 x$ are 1 and 2 respectivlely.
- The space of all polynomials of degree n or less with real coefficients:
  $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
  is $n + 1$ dimensional.

- What is the dimension of the space of all power series
  $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$?
- What is a power series?
- Give an example of a power series?

An example from mathematics $\cdots$ cont.

- What is the dimension of the space of all power series $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$?
- What is a power series?
- A power series (in one variable) is an infinite series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$ where a_n represents the coefficient of the nth term and c is a constant. a_n is independent of x and may be expressed as a function of n.

- What is the dimension the space of all power series
$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$?
- The space of all power series
$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$
is same as the space of an infinite sequence
$(a_0, a_1, a_2, \cdots, a_n, \cdots)$
which is infinite dimensional.

Simple Examples from Mathematician's Point of View

1. A point $x = 0$ on the real line is zero-dimensional space.
2. Real line is one-dimensional space. $0 \leq x_1 \leq 1$, a segment of a straight line of unit length is one-dimensional object (not space).
3. $x_1^2 + x_2^2 \leq 1$ is a circular region in a plane (a part of two dimensional space) is a 2-D object.
4. $x_1^2 + x_2^2 + x_3^2 \leq 1$ is a sphere in 3-dimensional space.
5. $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$ is a sphere in 4-dimensional space. Can we visualise it geometrically?

Mathematician Need Not Attempt to Visualise Geometrically

1. We may write for a 4-D object:
   modulus of the sum of the sequences of 4 real numbers is ≤ 1 or $|x_1 + x_2 + x_3 + x_4| \leq 1$
   but without thinking seriously in terms of hyperspace figures.
2. This is a four dimensional hyper cube. It it harder for you but try to see (start with $|x_1 + x_2| \leq 1$), then go to above object.
3. Scientifically, there is no evidence that the physical spaces of 4 or more dimensions exist.

Mathematician Need Not Attempt to Visualise Geometrically ··· cont.

4-dimensional relativistic space "space-time" is a mathematical concept. Relativity is difficult only when we try *to visualize its physically realistic results geometrically, which certainly does not exist.*

Let Us Attempt to Visualise n-D Objects Geometrically

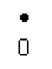
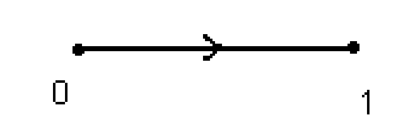
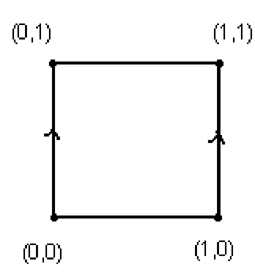


Figure 1. A point, $\{x_1 = 0\}$



2. Move point by the unit distance along a straight line to generate line segment, $\{0 \le x_1 \le 1\}$.

Let Us Attempt to Visualise n-D Objects Geometrically . . . . cont.



3. Move the line segment perpendicular to itself by the unit length to generates a square, $\{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$.

Let Us Attempt to Visualise n-D Space Geometrically cont......

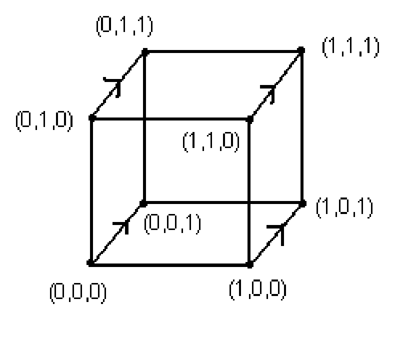


Figure 4. Shift the square in a direction right angle to its plane i.e., in x_3 diection to get a cube $\{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\}$

Let Us Attempt to Visualise n-D Space Geometrically cont......

- Our visual power ends there and we cannot proceed further.
- However, there is no logical reason why we can not assume that cube is shifted in a direction perpendicular to itself, i.e., in x_4 direction in (x_1, x_2, x_3, x_4)-space.

Shifting a Cube in Fourth Direction to Get a Tesseract

- If the cube is so shifted by unit distance, the object so generated is a unit hypercube, a tesseract in four dimensional space:
  $\{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1, 0 \leq x_4 \leq 1\}$.
- Since there is no 4-D, we can not visualise it. But note that we drew a projection of the cube on a plane.
- Let us project a tesseract on 2-D and draw on the paper as shown in Figure 5.

Shifting a Cube in Fourth Direction to Get a Tesseract

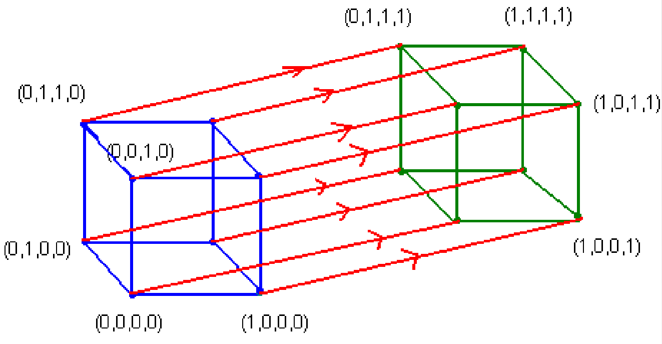


Figure 5. Tesseract is also called 8-cell or regular octachoron or cubic prism

Topologically Same and Different Objects

- In 2-D, a circle and a square are topologically same. You can bend and you can stretch, but you cannot break and in this process you can deform a circle into a square.
- The circle is topologically different from a figure 8, because although you can squash the middle of a circle together to make it into a figure 8 continuously, when you try to undo it, you have to break the connection in the middle and this is discontinuous: points that are all near the center of the eight end up split into two batches, on opposite sides of the circle, far apart.

Topologically Same Objects from Wikipedia

Topologically Different Objects

- We have drawn geometrical figures of a point, a line segment, a square and projections of a cube and tesseract in Euclidean spaces of dimensions 0, 1, 2, 3, 4 respectively.
- These objects are topologically distinct: a straight line can not be continuously deformed to a square*, a square deformed to a cube, a cube to a hypercube. ***Note:** Square means $\{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$, not just the boundary.
- Let us study some geometrical features of these objects.

Number of Corners and Edges

- **Number of corners :**
  A unit line: 2 end points.
  A square: $2 \times 2 = 4$ end points (corners).
  A cube: $2 \times 4 = 8$ corners.
  A tesseract: $2 \times 8 = 16$ corners.
- **Number of edges :**
  A unit line: 1 edge.
  A square: movement of unit line -2 from starting and end -2 from the lines created by the corners (end points) i.e., $2 \times 1 + 2 = 4$ edges.
  A cube: $2 \times 4 + 4 = 12$ edges.
  A tesseract: $2 \times 12 + 8 = 32$ edges.

- **Number of squares :**
  A square: 1 square.
  A cube: by movement of a square. 1 starting square, 1 end square and 4 squares made by the 4 edges of square i.e., $2 \times 1 + 4 = 6$.
  A tesseract: $2 \times 6 + 12 = 24$
- **Number of cubes :**
  A cube: 1 cube.
  A tessaract: $2 \times 1 + 6 = 8$ cubes.

A Table for Corners, Edges, Squares and Cubes

n-spaces	Points	Lines	Square	Cubes	Tesseracts
0	1	0	0	0	0
1	2	1	0	0	0
2	4	4	1	0	0
4	16	32	24	8	1
5	...	...	...	...	...

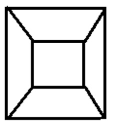
Formula for n space: Expand $(2x+1)^n$. The coefficients of powers of x give the number of elements.

$n = 4, \ (2x+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

This is an algorithm, a proof is required.

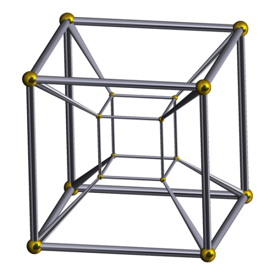
Projection of a cube on a 2-D

Projection of a cube on 2-D space due to a point light source when oriented properly.



- This projection shares all topological properties of the cube. See 12 lines and 6 squares.
- A fly can not walk along all edges of a cube in a continuous path without going over an edge twice nor can it do it on the above projected figure.

Projection of a tesseract on a 3-D:



- All elements of the tesseract can be identified.
- We have 8 cubes. Six cubes suffer projective distortions, they become 6 hexahedrons surrounding the small cube.

Simple Rotation of Tesseract from Wikipedia

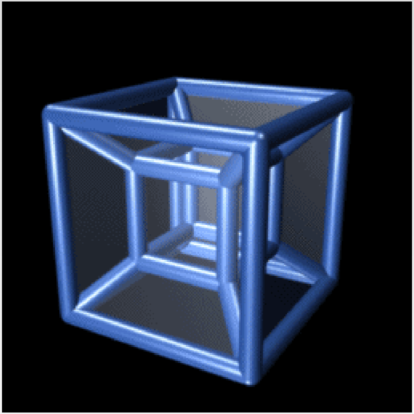


Figure: A 3D projection of a tesseract performing a simple rotation about a plane which bisects the figure from front-left to back-right and top to bottom

Double Rotation of Tesseract from Wikipedia

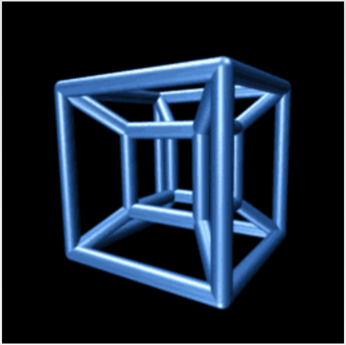
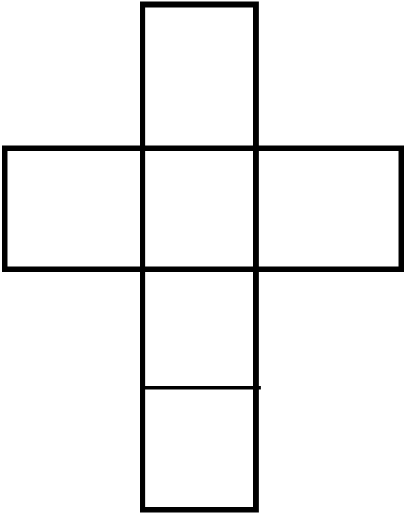


Figure: A 3D projection of a tesseract performing a double rotation about two orthogonal planes

Opening sides or Unfolding of Cube Resulting 6 Squares in 2-D



Opening or Unfolding of Tesseract Resulting 8 Cubes in 3-D

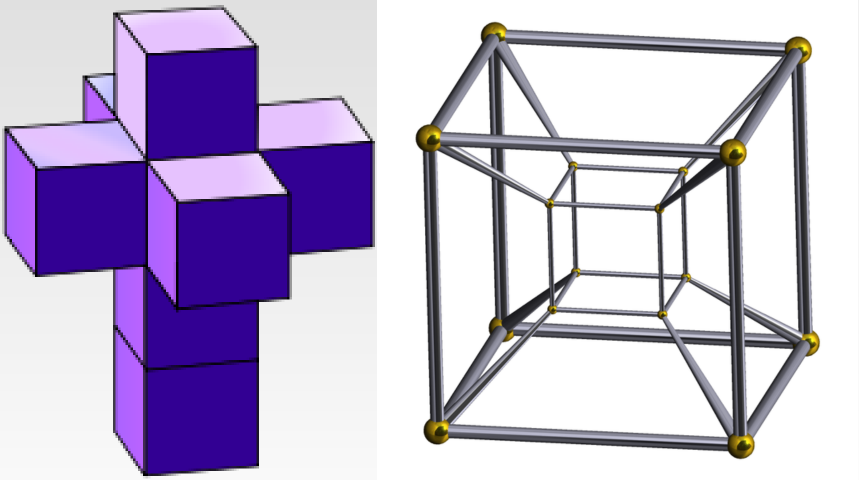


Figure: Unfolding of Tesseract from Wikipedia

- Suppose you are running very fast with a vertical pole and with a stone tied at the top of the pole.
- Suppose the stone suddenly falls. Where will it fall?
- Ptolemy (90-168) says it will fall behind you.
- Galileo (1564-1642) says it will fall at your feet.
- Who is right?
- To get the answer - which is very simple - see the case on the next slide.

- Suppose you are travelling in a fast moving train and you drop a stone from your your hand. Where will it fall?
- What is your answer?
- What will be the path seen by a person on the ground outside the train?
- Draw trajectory in both frames.
- Position P: in a coordinate system in a space is represented by (x, y, z).
- Event: The stone occupies different positions at different time. We can associate an event (x, y, z, t) at a point of the trajectory.

- Suppose you have two frames S and S'.
- Let S' moves with a constant velocity $\boldsymbol{v}$ with respect to S.
- The direction of coordinate axes is at our disposal. We choose x-axis and x'-axis in the direction of the relative velocity. Now $\boldsymbol{v} = (v, 0, 0)$
- Then our common experience give the Galilean transformation:
- $$x' = x - vt, \; y' = y, \; z' = z, \; t' = t. \tag{1}$$

Light trajectory for propagation in one D-space

- Draw trajectory of light starting from $x = 0$ at time $t = 0$ for propogation in 1-D space with constant velocity c of light.
- We wish to draw in (x, t)-plane.
- The trajectory is

$$x - ct = 0 \quad \text{and} \quad x + ct = 0.$$

- Both lines can be written in one equation

$$x^2 - c^2t^2 = 0$$

which we need to draw.

Light cone with constant velocity of light c

- Let us draw the trajectory of light starting from origin at $t = 0$ in (x_1, x_2, x_3, t)-space, assuming velocity of light to be constant.

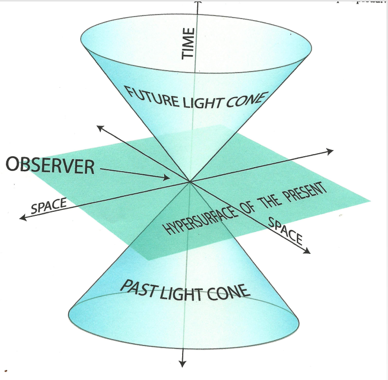


Figure: Equation is $x^2 + y^2 + z^2 - c^2t^2 = 0$. Interpret each part of this figure in 4-D: each section of the conoid is a sphere.

Light cone with constant velocity of light c - another figure

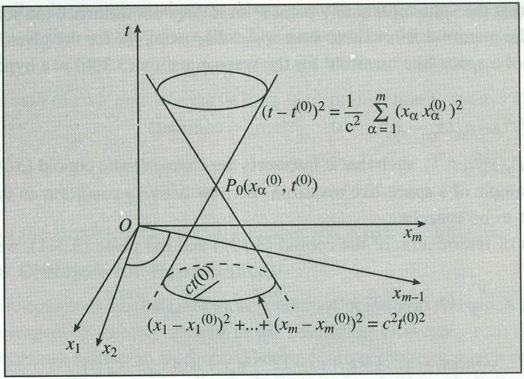


Fig. 3.21. *The lower and upper portions of the characteristic conoid through P_0 in space-time*

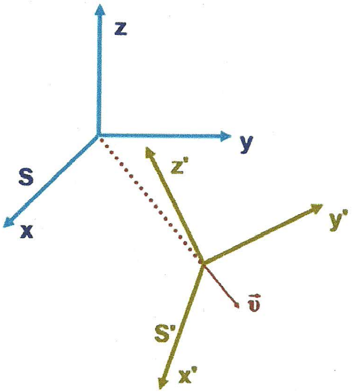
Figure: This is a generalisation in m-space dimensions.

Inertial Frames

- Under the Galilean transformation the equations of above light cone in (x, y, z, t)-space and (x', y', z', t')-space have different expressions.
- First see it for 1-D propagation. $x^2 - c^2t^2 = 0$ becomes $(x' + vt')^2 - c^2t'^2 = 0$.
- In 3-D space, conoid becomes $x^2 + y^2 + z^2 - c^2t^2 = (x' + vt')^2 + y^2 + z^2 - c^2t'^2 = 0$
- This, so obvious from our experience, speed of light should change.
- But it actually does not happen.
- This is because,the Galilean transformation is only approximately valid when the velocity $v << c$.

- Both, Newton and Einstein (Einstein for STR) talked about inertial frames
- All inertial frames are in a state of constant, rectilinear motion with respect to one another.
- Mathematically, transformation from one inertial frame to another is given by nonsingular linear transformation from (x, y, z, t) to (x', y', z', t').

Two Inertial Frames



Two inertial frames S and S'

The frame S' is rotated by an arbitrary but fixed rotation and moves with uniform velocity. Then the transformation between (x,y,z,t) and (x',y',z',t') is linear.

Mathematically, transformation from from one inertial frame to another is given by nonsingular linear transformation from (x, y, z, t) to (x', y', z', t').

- The most general, linear transformation between (x, t) and (x', t') is:

$$x' = a_1 x + a_2 t, \quad t' = b_1 x + b_2 t.$$

with constant a_1, a_2, b_1 and b_2 and $a_1 b_2 - a_2 b_1 \neq 0$.

- The two axioms of the *special theory of relativity* are:
  (i) Laws of physics are same in all inertial frames.
  (ii) The speed of light in free space has the same value c in all inertial frames.
- Axiom (ii), required for derivation of the transformation giving STR can be stated as:
  The light cone in space-time at $(0, 0, 0, 0)$ *is the same for all inertial frames.*
- We have seen, this is not so with Galilean transformation.

Special Theory of Relativity: Derivation Lorentz Transformation

- Therefore, invariance of the light cone, i.e., STR requires a linear transformation which comes from:

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \qquad (2)$$

- Assuming that $\boldsymbol{v}$ is in the direction of the x-axis, from the above condition, we can derive Lorentz transformation, which we write on the next slide.

- Lorentz Transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

$$y' = y \tag{4}$$

$$z' = z \tag{5}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6}$$

- As a particular case, when $v << c$, we get approximately the Galilean transformation

$$x' = x - vt, \; y' = y, \; z' = z, \; t' = t.$$

Invariance of Light Conoid Under Lorentz Transformation - Verification

- Lorentz transformation gives

$$x'^2 - c^2t'^2 = \frac{(x-vt)^2 - (t - \frac{v}{c^2}x)^2}{1-\frac{v^2}{c^2}} \tag{7}$$

$$= \frac{x^2(1-\frac{v^2}{c^2}) - c^2t^2(1-\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}} \tag{8}$$

$$= x^2 - c^2t^2. \tag{9}$$

- Beautiful result: Frames are in relative motion but conoid remains invariant in shape and position.

Space-like plane and time-like direction

In a space-like plane in (x, y, z, t)-space, light signals from one point P do not reach any other point. In the figure below R is space-like but T is not space-like.

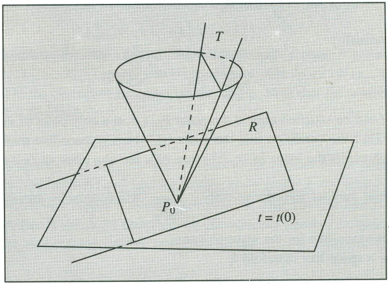


Fig. 3.22. *R is a space-like plane*

A time like direction points into the future light coniod or past light coniod or null cone.

Geometry of STR in one dimension

1. Draw it in (x, t)-plane.
2. x-axis means infinite velocity.
3. Time axis means zero velocity.
4. Movement along a generator of the light cone requires speed equal to the velocity of light.
5. Movement along a line in space-like plane requires velocity greater than the velocity of light.

Final Comment

- I have shown you there is no physical existence of 4-D space.
- But mathematicians do talk about n-D space and they can project figures on 2-D or 3-D, which you can visualize.
- The 4-D space of relativity is a mathematical space.

Reference

I was encouraged to prepare a part of this lecture in 1989 after reading a chapter in

Martin Gardner- Mathematical Carnival, Penguin Books.

Thank You

Thank you