TWO EXTENSION THEOREMS OF HARTOGS-CHIRKA TYPE INVOLVING CONTINUOUS MULTIFUNCTIONS

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1. Introduction and statement of results

This paper is motivated by a version of Hartogs' lemma that says that if Ω is some neighbourhood of the union of $\partial\mathbb{D} \times \mathbb{D}$ and a complex analytic subvariety $\Sigma \subset \overline{\mathbb{D}} \times \mathbb{D}$ that is finitely-sheeted over $\mathbb{D}$ (such that $\Omega \cap \mathbb{D}^2$ is connected), and $f \in \mathcal{O}(\Omega)$, then f continues holomorphically to $\mathbb{D}^2$; and by the Hartogs-type extension theorem of Chirka. Chirka's theorem says:

Result 1.1 (Chirka). *Let $\phi : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function having $\sup_{z \in \overline{\mathbb{D}}} |\phi(z)| < 1$ and let S be its graph. Let Ω be a connected open neighbourhood of $S \cup (\partial\mathbb{D} \times \mathbb{D})$ contained in $\{(z, w) \in \mathbb{C}^2 : |w| < 1\}$. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^2$.*

(Here, and in what follows, $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.) This motivates one to ask whether, given the following "Weierstrass pseudopolynomial"

$$\mathcal{P}_a(z, w) := w^k + \sum_{j=0}^{k-1} a_j(z) w^j, \;\; k \geq 2, \tag{1.1}$$

where $a_0, ..., a_{k-1} \in \mathcal{C}(\overline{\mathbb{D}})$, with $\mathcal{P}_a^{-1}\{0\} \subset \overline{\mathbb{D}} \times \mathbb{D}$, and a neighbourhood Ω of $\mathcal{P}_a^{-1}\{0\} \cup \big(\partial\mathbb{D} \times \overline{\mathbb{D}}\big)$, the conclusion of the aforementioned theorems can be inferred in this new setting.

One possible approach to this question is to investigate a version of Result 1.1 with one copy of $\mathbb{D}$ replaced by a bordered Riemann surface determined by $a := (a_0, ..., a_{k-1})$, over which the graph of the multifunction is transformed to a graph. One is then reduced to solving a certain quasilinear $\overline{\partial}$-problem analogous to the one considered by Chirka [4] (also see [5] by Chirka and Rosay). There is considerable literature on this subject; see, for instance, [7] by Koppelman. However, for this approach to work, one needs continuous dependence of solutions on the parameters, and *sup-norm* estimates with small norm, neither of which seem to be known at this time. A second approach is suggested by the Kontinuitätssatz-based strategies of Bharali [2] and Barrett-Bharali [1], provided one is willing to allow $(a_0, ..., a_{k-1})$ in (1.1) to belong to some strict subclass of $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$. To

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motivate the origins of the two main theorems below, let us state one of the results from [1] and [2].

Result 1.2 (Bharali, [2]). *Let Γ be the graph of the map $(\phi_1, ..., \phi_k) : \overline{\mathbb{D}} \to \mathbb{C}^k$, each $\phi_j(z) := \psi_j(z, \overline{z})$, where, for $j = 1, ..., k$,*

$$\psi_j \in \left\{ \psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(z,\zeta)\in\mathbb{D}^2} |\psi(z,\zeta)| < 1 \text{ and } z \mapsto \psi(z, \overline{z}) \text{ is continuous on } \overline{\mathbb{D}} \right\}. \tag{1.2}$$

If Ω is a connected neighbourhood of $S := \Gamma \cup (\partial\mathbb{D} \times \mathbb{D}^k)$ contained in $\{(z, w) \in \mathbb{C} \times \mathbb{C}^k : w \in \mathbb{D}^k\}$ and if $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^{k+1}$.

In the theorems in [1] and [2], the authors construct a continuous family of discs $\{\Phi_t \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k) : t \in [0, 1]\}$ such that $\Phi_0 = (\phi_1, ..., \phi_k)$ and each Φ_t is holomorphic on larger and larger sub-regions of $\mathbb{D}$ so that, eventually, $\Phi_1 \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. This suggests the following strategy:

- *Step 1.* Setting $(\phi_1, ..., \phi_k) := (a_0, ..., a_{k-1})$, we can try to construct a continuous family of discs $\{\Phi_t\}_{t\in[0,1]}$ with the properties mentioned above. We can then treat each $\Phi_t := (\Phi_{t,0}, ..., \Phi_{t,k-1})$ as a k-tuple of the ordered coefficients of a Weierstrass pseudopolynomial, to obtain a continuous family of "pseudovarieties" $\left\{\Sigma_t := \left\{(z, w) \in \mathbb{D} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} \Phi_{t,j}(z)w^j = 0\right\}\right\}_{t\in[0,1]}$ such that $\Sigma_0 := \{(z, w) \in \overline{\mathbb{D}} \times \mathbb{C} : \mathcal{P}_a(z, w) = 0\}$, each Σ_t is a finitely-sheeted complex analytic subvariety fibered over larger and larger sub-regions of $\mathbb{D}$, and Σ_1 is the graph of an analytic multifunction (i.e., a multigraph) over $\mathbb{D}$.
- *Step 2.* In the above construction, our hypotheses on $(a_0, ..., a_{k-1})$ must also ensure that each Σ_t over $\mathbb{D}$, like the initial "pseudovariety", lies within the bidisc, i.e., $\Sigma_t \subset \overline{\mathbb{D}} \times \mathbb{D}\ \forall t \in [0, 1]$, and that Σ_t is attached to $\partial\mathbb{D} \times \mathbb{D}$ along the border of $\Sigma_t\ \forall t \in [0, 1]$.
- *Step 3.* Finally, we invoke a suitable version of the Kontinuitätssatz to achieve analytic continuation along the family constructed above so as to reduce the problem to the finitely-sheeted-analytic-variety version of Hartogs' lemma mentioned in the beginning of this section.

It turns out that this second strategy is successful (with some refinement) if the coefficients $a_0, ..., a_{k-1}$ are drawn from the subclasses studied in [1] and [2]. Now, one may ask why Step 1 of the above strategy cannot be attempted for $a_0, \ldots, a_{k-1} \in \mathcal{C}(\overline{\mathbb{D}})$. But this would amount to attempting to prove a vector-valued version of Chirka's main result (Result 1.1) in [4]. But, Rosay's counterexample in [9] establishes that Chirka's result cannot be generalized to higher dimensions in its entire generality — i.e. when $a_0, ..., a_{k-1}$, $k > 1$, are merely continuous. The first theorem of this paper is stated for $a_0, ..., a_{k-1}$ belonging to the subclass of $\mathcal{C}(\overline{\mathbb{D}})$ introduced by Barrett and Bharali in [1].

Theorem 1.3. *Let $a_0, ..., a_{k-1} \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C})$ be such that the set*

$$\Sigma_a := \left\{ (z,w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z) w^j = 0 \right\}$$

lies entirely in $\overline{\mathbb{D}} \times \mathbb{D}$. For $0 < r \leq 1$, let $A^j_\nu(r)$ represent the ν^{th} Fourier coefficient of $a_j(re^{i\cdot})$, $\nu \in \mathbb{Z}$. Assume that $A^j_\nu \equiv 0$ $\forall \nu < 0$ and $j = 0, ..., k-1$. Let Ω be a connected neighbourhood of $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$ such that $\Omega \cap \mathbb{D}^2$ is connected. Then, for every $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega\cap\mathbb{D}^2} \equiv f\big|_{\Omega\cap\mathbb{D}^2}.$$

Our next theorem has its origins in Result 1.2, but see Remarks 1.5 and 1.6 below.

Theorem 1.4. *Let $a_j := \psi_j(z, \overline{z})$, where*
(1.3)
$$\psi_j \in \left\{ \psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(\zeta,s)\in\mathbb{D}\times[0,1]} |\psi(\zeta, s\overline{\zeta})| < 1 \text{and } z \mapsto \psi(z,\overline{z}) \text{ is continuous on } \overline{\mathbb{D}} \right\}$$

for $j = 0, ..., k-1$, be such that the set

$$\Sigma_a := \left\{ (z,w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z) w^j = 0 \right\}$$

lies entirely in $\overline{\mathbb{D}} \times D(0;2)$. Let Ω be a connected neighbourhood of $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$ such that $\Omega \cap \big(\mathbb{D} \times D(0;2)\big)$ is connected. Then, for every $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\mathbb{D} \times D(0;2))$ such that

$$F\big|_{\Omega\cap\left(\mathbb{D}\times D(0;2)\right)} \equiv f\big|_{\Omega\cap\left(\mathbb{D}\times D(0;2)\right)}.$$

Remark 1.5. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the classes of functions appearing in (1.2) and (1.3) respectively. While Theorem 1.4 stems from Result 1.2, it must be admitted that the class $\mathfrak{F}_1$ is quite restrictive. However, while adapting the approach outlined above, we found that we could construct the deformation $\{\Phi_t : t \in [0,1]\}$ in a slightly different fashion from what is suggested in [2], which allows us to work with $a_0, ..., a_{k-1}$ belonging to a less restrictive class. Note that $\mathfrak{F}_2 \supsetneq \mathfrak{F}_1$; simply observe that if $\psi(z,w) := (M+\varepsilon)^{-1}\exp(z - w - 2)$, where $M = \sup_{(\zeta,s)\in\overline{\mathbb{D}}\times[0,1]} |\exp(\zeta - s\overline{\zeta} - 2)|$, then $M < 1$ and for $\varepsilon \in (M, 1)$, $\psi \in \mathfrak{F}_2$ but $\psi \notin \mathfrak{F}_1$.

Remark 1.6. Unbeknownst to me, Černe and Flores [3] have independently used the three-step method summarized earlier to prove:

($*$) *Let $a_0, ..., a_{k-1}$ be continuous functions on $\overline{\mathbb{D}}$ and let*

$$\Sigma_a := \{(z,w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + a_{k-1}(z) w^{k-1} + \cdots + a_0(z) = 0\}$$

be a continuous variety over $\overline{\mathbb{D}}$. Then, every function holomorphic in a connected neighbourhood of the set $S = \Sigma_a \cup (\partial\mathbb{D} \times \mathbb{C})$ extends holomorphically to a neighbourhood of $\overline{\mathbb{D}} \times \mathbb{C}$.

Note that $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{D})$ is a subset of the uniform closure (on $\overline{\mathbb{D}}$) of the function space obtained if we drop the bound "$\sup_{(z,\zeta)\in\mathbb{D}^2} |\psi(z,\zeta)| < 1$" from $\mathfrak{F}_1$. It is this, coupled with their reliance on the three-step method outlined above, that compels Černe-Flores to work with the *unbounded cylinder* $\overline{\mathbb{D}} \times \mathbb{C}$. Theorem 1.4 represents an alternative setting in which to exploit the same method with — in contrast to Černe-Flores [3] — the following initial objectives:

- to use the ideas of Barrett and Bharali to demonstrate an analytic-continuation theorem stated for a *compact* Hartogs figure ($S = \Sigma_a \cup \big(\partial\mathbb{D} \times \overline{D(0;2)}\big)$ in our case); and
- to extend the applicability of Result 1.2 to a less restrictive class of graphs/coefficients, namely $\mathfrak{F}_2$.

Due to considerations inherent to the three-step method we intend to use — see Remark 2.3(i) below — we, just like Černe-Flores, cannot work with the Hartogs configuration $\Sigma_a \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$ either. However, we *can* state a result involving $\Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$.

Many of the mathematical details underlying Step 2 and 3 are common to Theorems 1.3 and 1.4. These technicalities have been collected in Section 2. The actual proofs of Theorems 1.3 and 1.4 are presented in Sections 3 and 4 respectively.

2. Preliminary Lemmas

We will first isolate the technical elements of the two main proofs in the form of a few preliminary results. The following notation will be used:

- $D(a; r)$ will denote the open disc of radius r with centre at a, and $Ann(a; r, R)$ will denote the open annulus with centre at $a \in \mathbb{C}$ and having inner radius and outer radius r and R respectively;
- $\mathcal{C}^\infty(\overline{\mathbb{D}}; \mathbb{C})$ will denote the class of infinitely differentiable functions on the unit disc, all of whose derivatives extend to functions in $\mathcal{C}(\overline{\mathbb{D}})$;
- for $\alpha := (\alpha_0, ..., \alpha_{k-1}) \in \mathcal{C}(\overline{G}; \mathbb{C}^k)$, $k \in \mathbb{N}$, $G \subset \mathbb{C}$ a bounded domain, and $E \subset \overline{G}$
$$\mathcal{P}_\alpha(z, w) := w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j,$$
$$\Sigma_{\alpha,E} := \left\{ (z, w) \in E \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j = 0 \right\},$$
and, for the sake of convenience, the subscript E shall be dropped when $E = \overline{\mathbb{D}}$, i.e., $\Sigma_{\alpha,\overline{\mathbb{D}}} =: \Sigma_\alpha$.

The first step of the three-step strategy outlined in Section 1 is not difficult, but the details involved are theorem-specific. This is, in part, due to the requirements described in Step 2. The task of determining sufficient, yet not too strong, conditions on the coefficient k-tuple $(a_0, ..., a_{k-1})$ that will enable us to establish that each $\Sigma_t,\ t \in [0,1]$, is contained in the bidisc relevant to each theorem is a crucial one. The following lemma — a maximum principle for varieties — will prove useful.

Lemma 2.1. *Let $G \subset \mathbb{C}$ be a bounded domain and $a \in \mathcal{O}(G;\mathbb{C}^k) \cap \mathcal{C}(\overline{G};\mathbb{C}^k)$. Define*

$$M(z) := \max\left\{|w| : (z,w) \in \Sigma_{a,\overline{G}}\right\}.$$

If $M(z) \leq K\ \forall z \in \partial G$, then $M(z) \leq K\ \forall z \in \overline{G}$.

Proof. We would be done if we could obtain the conclusion of this lemma when $\Sigma_{a,G}$ is an irreducible subvariety. For $\Sigma_{a,G}$ irreducible, if we can show that M is subharmonic, then the result would follow from the maximum principle.

Recall that the zeros of monic degree-k polynomials over $\mathbb{C}$, viewed as *unordered* k-tuples of zeros repeated according to multiplicity, vary continuously with the coefficients. Hence, as M is symmetric in the zeros of $\mathcal{P}_a$, $M \in \mathcal{C}(\overline{G})$.

Now, let

$$\mathfrak{R}(z) := \text{ resultant of } \mathcal{P}_a(z,\bullet) \text{ and } \partial_w \mathcal{P}_a(z,\bullet),\ z \in G.$$

By the irreducibility of $\Sigma_{a,G}$, $\mathfrak{R} \not\equiv 0$. As $\mathfrak{R} \in \mathcal{O}(G)$, $\mathfrak{S} := \mathfrak{R}^{-1}\{0\}$ is a discrete set in G. Now, for any $z_0 \in G \setminus \mathfrak{S}$, $\Sigma_{a,\{z_0\}} = \{(z_0, w_{0,1}), ..., (z_0, w_{0,k})\}$ with $w_{0,j} \neq w_{0,l}$ for $j \neq l$. As $\partial_w \mathcal{P}_a(z_0, w_{0,j}) \neq 0$ for each $j = 1, ..., k$, we may apply the implicit function theorem at each point of $\Sigma_{a,\{z_0\}}$ to obtain a common radius $r(z_0) > 0$ such that the k sheets of $\Sigma_{a,D(z_0;r(z_0))}$ are the graphs of functions $\phi_1^{z_0}, ..., \phi_k^{z_0} \in \mathcal{O}\big(D\,(z_0; r(z_0))\,\big)$. Clearly,

$$M(z) = \max_{j \leq k} \left|\phi_j^{z_0}(z)\right| \quad \forall z \in D\big(z_0; r(z_0)\big).$$

Thus, $M\big|_{D(z_0;r(z_0))}$ is subharmonic. As z_0 was arbitrarily chosen from the open set $G\setminus\mathfrak{S}$, we infer that $M\big|_{G\setminus\mathfrak{S}}$ is a subharmonic function.

As $\mathfrak{S}$ is the zero set of a holomorphic function, it is a polar set. But $M\big|_{G\setminus\mathfrak{S}}$ is a bounded subharmonic function, and $M \in \mathcal{C}(\overline{G})$. Therefore, M must be subharmonic in G [8, p. 47]. $\square$

Remark 2.2. The following is a paraphrasing of the above lemma that will be used in our situation.

Let $G \subset \mathbb{C}$ be a bounded domain and $a \in \mathcal{O}(G;\mathbb{C}^k) \cap \mathcal{C}(\overline{G};\mathbb{C}^k)$. Then,

$$\Sigma_{a,\partial G} \subset \partial G \times D(0;K) \Rightarrow \Sigma_{a,\overline{G}} \subset \overline{G} \times D(0;K).$$

Remark 2.3. We will also need the following algebraic facts:

(i) If $\alpha_0, ..., \alpha_{k-1} \in \mathbb{D}$, $k \in \mathbb{N}$, and $w_1, ..., w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w^1 + \alpha_0$, then $w_j \in D(0;2)$, $j = 1, ..., k$. For an easy proof of this fact, one can apply Rouché's theorem to $f(w) := w^k$ and $g(w) := w^k + \sum_{j=0}^{k-1} \alpha_j w^j$ on $\partial D(0;2)$.

(ii) If $(\alpha_0, ..., \alpha_{k-1}) \in \mathbb{C}^k$, and $w_1, ..., w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w + \alpha_0$, then, for $\eta \in \mathbb{C}$, $w_1 + \eta, ..., w_k + \eta$ are the zeros of the polynomial

$$w^k + \alpha_{k-1}^{(\eta)} w^{k-1} + \cdots + \alpha_1^{(\eta)} w + \alpha_0^{(\eta)},$$

where, for each j,

$$\alpha_j^{(\eta)} = \alpha_j + \sum_{l=j+1}^{k} (-1)^{l-j} \binom{l}{l-j} \alpha_l \eta^{l-j} \tag{2.1}$$

interepreting $\alpha_k := 1$.

Theorems having a similar flavour as Theorems 1.3 and 1.4 have relied upon the Kontinuitätssatz. However, the earliest (and partially correct) works do not specify *which* form of the "Kontinuitätssatz" they rely upon. We wish, here, to make clear that the version that works for us is the version of Chirka and Stout [6]. However, merely using the Chirka-Stout Kontinuitätssatz will yield a conclusion weaker than desired, on the envelope of holomorphy of the domain in question. The next lemma follows the approach of Barrett and Bharali [1] to argue that it is, in fact, possible to obtain the strong conclusion of Chirka's extension theorem (i.e. Result 1.1) [4] in our situation.

Lemma 2.4. *Let $a = (a_0, ..., a_{k-1}) \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ and $\Sigma_a \subset \overline{\mathbb{D}} \times D(0;r)$, $r > 0$. Let Ω be a connected open neighbourhood of $S := \Sigma_a \cup \big(\partial\mathbb{D} \times \overline{D(0;r)}\big)$ and $f \in \mathcal{O}(\Omega)$. Let $V := Ann(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, be such that $V \times D(0;r) \subset \Omega$, and let $D \Subset \Omega$, be an open subset containing S. For any $\alpha \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ and any $\eta \in \mathbb{C}$, let $\alpha^{(\eta)} \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ denote the perturbation that is given by (2.1) so that $\Sigma_{\alpha^{(\eta)}} = \Sigma_\alpha + (0, \eta)$. Suppose there exists a continuous function $A := (A_0, ..., A_{k-1})$ on $\overline{\mathbb{D}} \times [0,1]$ such that $A(\cdot, 0) = a(\cdot)$, and a $\delta > 0$ which is so small that, defining $\Sigma_t^\eta := \Sigma_{A^{(\eta)}(\cdot, t)}$, we have*

(1) *for each $\eta \in D(0;\delta)$, $\Sigma_t^\eta \subset \overline{\mathbb{D}} \times D(0;r)$ $\forall t \in [0,1]$; and*

(2) *for each $\eta \in D(0;\delta)$, $\Sigma_t^\eta \cap (\mathbb{D} \times D(0;r)) \setminus \overline{D}$ is a complex-analytic subvariety of $\mathbb{D} \times D(0;r) \setminus \overline{D}$.*

Then, there exists a connected neighbourhood Ω_1 of $S_1 := \Sigma_1^0 \cup \big(\partial\mathbb{D} \times D(0;r)\big)$ and $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)}.$$

Proof. Let

$$\mathcal{T} := \bigcup_{\eta \in D(0;\delta)} \Sigma_1^0 + (0,\eta).$$

By the Chirka-Stout Kontinuitätssatz [6], $\mathcal{T} \subset \pi(\widetilde{\Omega})$, where $(\widetilde{\Omega}, \pi)$ denotes the envelope of holomorphy of Ω.

There is a canonical holomorphic imbedding of Ω into $\widetilde{\Omega}$. We denote this imbedding by $j : \Omega \hookrightarrow \widetilde{\Omega}$. Corresponding to each $f \in \mathcal{O}(\Omega)$, there is a holomorphic function $\mathcal{E}(f) \in \mathcal{O}(\widetilde{\Omega})$ such that $\mathcal{E}(f) \circ j = f$. By [6] (and analogous to the situation in [1]), there exists a holomorphic mapping (note that Σ_1^η varies analytically in η) $H : \mathcal{T} \to \widetilde{\Omega}$ such that

$$\pi \circ H\big(\Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w)\big) = \Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w) \quad \forall \eta \in D(0;\delta) \text{ and } z \in \overline{\mathbb{D}}.$$

Now, for each $p := (z_1, w_1) \in \mathcal{T} \cap \big(V \times D(0;r)\big)$, there exist

- an $\eta_0 \in D(0;\delta)$; and
- a point $q \in \Sigma_0^{\eta_0} \cap \{z_1\} \times \mathbb{C}_w$,

such that the continuous family $\{\Sigma_t^{\eta_0}\}_{t \in [0,1]}$ determines a path $\gamma_{qp} : [0,1] \to \{z_1\} \times \mathbb{C}_w$ with $\gamma_{qp}(0) = q$ and $\gamma_{qp}(1) = p$. Let $\mathfrak{S}_\Omega :=$ the sheaf of $\mathcal{O}(\Omega)$-germs over $\mathbb{C}^2$ (refer to [8, Chapter 6] for the definition of an $\mathcal{O}(\Omega)$-germ) and let

$$\widetilde{\gamma_{qp}} := \text{the lift of } \gamma_{qp} \text{ to } \mathfrak{S}_\Omega \text{ starting at the germ } [g : g \in \mathcal{O}(\Omega)]_q.$$

Examining the Kontinuitätssatz, $H(p) = \widetilde{\gamma_{qp}}(1)$.

We know that if $[s_g : g \in \mathcal{O}(\Omega)]_z$ is an $\mathcal{O}(\Omega)$-germ in $\widetilde{\Omega}$, then

$$\mathcal{E}(f)\left([s_g : g \in \mathcal{O}(\Omega)]_z\right) = s_f(z).$$

By the monodromy theorem, $\widetilde{\gamma_{qp}}(1) = [g : g \in \mathcal{O}(\Omega)]_p$. Hence,

$$\mathcal{E}(f) \circ H(p) = \mathcal{E}(f)\left(\widetilde{\gamma_{qp}}(1)\right) = f(p).$$

Since the above holds for any arbitrary $p \in \mathcal{T} \cap \big(V \times \ D(0;r)\big)$, we see that

$$\mathcal{E}(f) \circ H = f \text{ on } \mathcal{T} \cap \big(V \times \ D(0;r)\big).$$

Finally, let $\Omega_1 := \mathcal{T} \cup (V \times D(0;r))$ and

$$f_1(z,w) := \begin{cases} \mathcal{E}(f) \circ H(z,w), & \text{if } (z,w) \in \mathcal{T}, \\ f(z,w), & \text{if } (z,w) \in V \times D(0;r), \end{cases}$$

Then, $f_1 \in \mathcal{O}(\Omega_1)$ and

$$f_1\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)}.$$

$\square$

3. Proof of Theorem 1.3

By Lemma 5 in the paper [1] by Barrett and Bharali, and the continuous dependence of the zeros of a polynomial on its coefficients, we know that it is enough to prove Theorem 1.3 for $a_0, ..., a_{k-1} \in \mathfrak{G}_1$, where $\mathfrak{G}_1 \subsetneq \mathcal{C}(\overline{\mathbb{D}};\mathbb{C})$ is the following set:

$$\left\{ g \in \mathcal{C}^\infty(\overline{\mathbb{D}};\mathbb{C}) : \exists N \in \mathbb{N},\ G_n \in \mathcal{C}^\infty\big([0,1];\mathbb{C}\big) \text{ such that } g(re^{i\theta}) = \sum_{n=0}^{N} G_n(r)e^{in\theta},\ r \in (0,1] \right\}.$$

Thus, we replace $a = (a_0, ..., a_{k-1})$ in Theorem 1.3 by $b := (b_0, ..., b_{k-1}) \in \mathfrak{G}_1^k$. This is because we can find a Σ_b that is so close to Σ_a that $\Sigma_b \subset \Omega$ and is attached to $\partial\mathbb{D} \times \overline{\mathbb{D}}$.

Fix a $j \in \{0, ..., k-1\}$. Let

$$b_j(re^{i\theta}) = \sum_{n=0}^{n(j)} B_n^j(r)e^{in\theta},\ \theta \in [0, 2\pi),$$

where $n(j) \in \mathbb{N}$ and $B_n^j \in \mathcal{C}^\infty\big([0,1];\mathbb{C}\big)$. Using Lemma 3 in [1], where Barrett and Bharali constructed an explicit family of analytic discs in $\overline{\mathbb{D}} \times \mathbb{C}$ with boundaries in $\{(z, b_0(z), ..., b_{k-1}(z)) : z \in \mathbb{D}\}$, we define a family of continuous discs $\{\mathfrak{B}_t = (\mathfrak{B}_{t,0}, ..., \mathfrak{B}_{t,k-1})\}_{t\in[0,1]}$ as follows:

$$\mathfrak{B}_{t,j}(\zeta) := \begin{cases} \sum\limits_{n=0}^{n(j)} B_n^j(t)\left(\frac{\zeta}{t}\right)^n, & \text{if } \zeta \in D(0;t), \\ \\ b_j(\zeta), & \text{if } \zeta \in \overline{Ann(0;t,1)}. \end{cases} \tag{3.1}$$

Note that $\mathfrak{B}_0 = b$. Also, by Lemma 4 in [1], $\{\mathfrak{B}_t\}_{t\in[0,1]}$ is a continuous family, and $\mathfrak{B}_1 \in \mathcal{O}(\mathbb{D};\mathbb{C}^k) \cap \mathcal{C}(\overline{\mathbb{D}};\mathbb{C}^k)$.

Let $\delta > 0$ be so small that $\eta \in D(0;\delta) \Rightarrow \Sigma_b + (0,\eta) \subset \Omega \cap \big(\overline{\mathbb{D}} \times \mathbb{D}\big)$. Let $b^{(\eta)} = \left(b_1^{(\eta)}, ..., b_{k-1}^{(\eta)}\right)$ be defined pointwise by (2.1) in Section 2. By Remark 2.3(ii), each $b_j^{(\eta)}$, being a linear combination of $b_j, ..., b_{k-1}$, is in $\mathfrak{G}_1$. Thus, we can define continuous discs $\left\{\mathfrak{B}_t^{(\eta)} = \left(\mathfrak{B}_{t,0}^{(\eta)}, ..., \mathfrak{B}_{t,k-1}^{(\eta)}\right)\right\}_{t\in[0,1]}$ using the Fourier coefficients of $b_j^{(\eta)}(re^{i\cdot})$, $r \in (0,1]$, just as in equation (3.1). It is a simple observation that the same discs can be obtained by defining, on $\overline{\mathbb{D}}$,

$$\mathfrak{B}_{t,j}^{(\eta)} := \mathfrak{B}_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j}\binom{l}{l-j}\mathfrak{B}_{t,l}\eta^{l-j} + (-1)^{k-j}\binom{k}{k-j}\eta^{k-j}. \tag{3.2}$$

It is important to note that $\mathfrak{B}_t^{(0)} \equiv \mathfrak{B}_t\ \forall t \in [0,1]$.

Fix a domain $D \Subset \Omega$, such that $S \subset D$. We claim that the continuous family $\left\{\mathfrak{B}_t^{(\eta)}\right\}_{t\in[0,1]}$ satisfies the following properties:

a) $\mathfrak{B}_0^{(\eta)} = b^{(\eta)}\ \forall \eta \in D(0;\delta)$;

b) for a fixed t, $\mathfrak{B}_t^{(\eta)}$ depends analytically on η;

c) for each $\mathfrak{B}_t^{(\eta)}$, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \setminus \overline{D}$ is an analytic subvariety of $\overline{\mathbb{D}} \times \mathbb{C} \setminus \overline{D}$; and

d) for each t, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \subset \overline{\mathbb{D}} \times \mathbb{D}\ \forall \eta \in D(0;\delta)$.

Properties a) and b) follow from construction. For c), it is enough to observe that

$$\Sigma_{\mathfrak{B}_t^{(\eta)}} = \left(\Sigma_{b^{(\eta)},\overline{Ann(0;t,1)}}\right) \bigcup \left(\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)}\right),$$

and that $\mathfrak{B}_t^{(\eta)}\big|_{D(0;t)} \in \mathcal{O}(D(0;t);\mathbb{C}^k)$. For property d), it is enough to show that

$$\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)} \subset \overline{\mathbb{D}} \times \mathbb{D}.$$

But this follows from Lemma 2.1 applied to $\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)}$, with $D(0;t)$ acting as G, since

$$\Sigma_{\mathfrak{B}_t^{(\eta)},\partial D(0;t)} \equiv \Sigma_{b^{(\eta)},\partial D(0;t)} \subset \partial D(0;t) \times \mathbb{D}.$$

From this, we can conclude that the mapping $A : \overline{\mathbb{D}} \times [0,1] \to \mathbb{C}^k$ with $A(z,t) := \mathfrak{B}_t(z)$ satisfies the hypotheses of Lemma 2.4. Thus, there exists a connected open neighbourhood Ω_1 of $S_1 := \Sigma_{\mathfrak{B}_1^{(0)}} \cup \big(\partial\mathbb{D} \times \mathbb{D}\big)$ and a $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1\big|_{\Omega_1 \cap \left(V \times \mathbb{D}\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times \mathbb{D}\right)},$$

where $V := Ann(0; 1-\varepsilon, 1+\varepsilon),\ \varepsilon > 0$, such that $V \times \mathbb{D} \subset \Omega$.

But, $\mathfrak{B}_1^{(0)}$ is holomorphic by construction. Hence, in view of the ideas presented in Remark 2.3, we can argue as follows:

- Let $\{\mathfrak{U}_s\}_{s\in[0,1]}$ be defined as follows:

  $$\mathfrak{U}_s = (\mathfrak{U}_{s,0}, ..., \mathfrak{U}_{s,k-1}) := (s^k\mathfrak{B}_{1,0}^{(0)}, ..., s^{k-j}\mathfrak{B}_{1,j}^{(0)}, ..., s\mathfrak{B}_{1,k-1}^{(0)}).$$

  Then, for each $s \in [0,1]$, $\mathfrak{U}_s$ is analytic and $\Sigma_{\mathfrak{U}_s} = \left\{(z, sw) : (z\, w) \in \Sigma_{\mathfrak{B}_1^{(0)}}\right\}$.
- Let $\widetilde{\delta} > 0$ be so small that $\eta \in D(0;\widetilde{\delta}) \Rightarrow \Sigma_{\mathfrak{U}_1} + (0,\eta) \subset \Omega_1 \cap (\overline{\mathbb{D}} \times \mathbb{D})$. Note that, by Remark 2.3(ii), each $\mathfrak{U}_s^{(\eta)}$ depends analytically on η.
- We are now in a position to apply Lemma 2.4 to the continuous mapping $U : \overline{\mathbb{D}} \times [0,1] \to \mathbb{C}^k$ with $U(z,s) := \mathfrak{U}_{1-s}(z)$. Thus, there exists a connected open neighbourhood Ω_2 of $S_2 := \Sigma_{\mathfrak{U}_0} \cup \big(\partial\mathbb{D} \times \mathbb{D}\big) = (\overline{\mathbb{D}} \times \{0\}) \cup \big(\partial\mathbb{D} \times \mathbb{D}\big)$ and a $f_2 \in \mathcal{O}(\Omega_2)$ such that

  $$f_2\big|_{\Omega_2 \cap \left(V \times \mathbb{D}\right)} \equiv f_1\big|_{\Omega_2 \cap \left(V \times \mathbb{D}\right)},$$

  where $V := Ann(0; 1-\varepsilon, 1+\varepsilon),\ \varepsilon > 0$, such that $V \times \mathbb{D} \subset \Omega$.

By the classical theorem of Hartogs, $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega_2 \cap \mathbb{D}^2} \equiv f_2\big|_{\Omega_2 \cap \mathbb{D}^2}.$$

Thus, F and f must coincide in $\Omega_2 \cap \Omega_1 \cap \big(V \times \mathbb{D}\big) \cap \mathbb{D}^2$. As the latter is an open subset of the connected set $\Omega \cap \mathbb{D}^2$,we conclude that

$$F\big|_{\Omega\cap\mathbb{D}^2} \equiv f\big|_{\Omega\cap\mathbb{D}^2}.$$

$\square$

4. Proof of Theorem 1.4

The proof of this theorem is similar to that of Theorem 1.3. The main difference lies in the specific method of constructing, starting from the given multigraph, a continuous family of multigraphs along which we can achieve analytic continuation by invoking the Kontinuitätssatz. Recall that, in Section 3, the form of each coefficient function a_j facilitated the construction of functions that were holomorphic on increasing concentric discs in $\mathbb{D}$. In the present case, to perturb the coefficients, we will construct analytic annuli attached to the graphs of a_j along their inner boundaries, and to $\partial\mathbb{D} \times \mathbb{D}$ along their outer boundaries. In view of Remark 2.3(i), we are compelled to work with a polydisc longer than $\mathbb{D}^2$.

4.1. **The proof of Theorem 1.4.** Let $a(z) = \psi(z, \overline{z}) := (\psi_0(z, \overline{z}), ..., \psi_{k-1}(z, \overline{z}))$. Set $\mathcal{R} := \overline{\mathbb{D}} \times [0, 1]$. Note that, by hypothesis, we can find an $\varepsilon > 0$ such that $Ann(0; 1 - \varepsilon, 1 + \varepsilon) \times \mathbb{D}(0; 2) \subset \Omega$. Hence, (keeping in mind the closing arguments in Section 3) it suffices to work with $\Sigma_{a,\overline{D(0;1-\varepsilon/2)}}$ and the Hartogs configuration $S_\varepsilon := \Sigma_{a,\overline{D(0;1-\varepsilon/2)}} \cup \big(\partial D(0; 1 - \varepsilon/2) \times \overline{D(0; 2)}\big)$. This affords us the very useful property:

$$(\zeta, s) \mapsto \psi_j(\zeta, s\overline{\zeta}) \text{ is continuous on } \overline{D(0; 1 - \varepsilon/2)} \times [0, 1],\ \forall j = 0, ..., k - 1.$$

Therefore, it actually suffices to prove Theorem 1.4 under the assumption that $\psi_0, ..., \psi_{k-1} \in \mathfrak{G}_2$, where

$$\mathfrak{G}_2 := \left\{ \psi \in \mathcal{O}(\mathbb{D}^2) \cap \mathcal{C}(\overline{\mathbb{D}}^2) : \sup_{(\zeta,s)\in\mathcal{R}} |\psi(\zeta, s\overline{\zeta})| < 1 \right\}.$$

In order to avoid messy subscripted notation such as $\Sigma_{a,\overline{D(0;1-\varepsilon/2)}}$ and messy normalizations, we shall hereafter assume that $\psi_j \in \mathfrak{G}_2$, for $j = 0, ..., k - 1$.

We define a family of continuous discs $\{\Psi_t = (\Psi_{t,0}, ..., \Psi_{t,k-1})\}_{t\in[0,1)}$ as follows:

$$\Psi_t(\zeta) := \begin{cases} a(\zeta) = \psi(\zeta, \overline{\zeta}), & \text{if } \zeta \in D(0; 1 - t), \\ \\ \psi\left(\zeta, \frac{(1-t)^2}{\zeta}\right), & \text{if } \zeta \in \overline{Ann(0; 1 - t, 1)}. \end{cases} \tag{4.1}$$

Therefore, $\Psi_0 = a$. We observe that $\{\Psi_t\}_{t\in[0,1)}$ is a continuous family in the sense that for a fixed $\zeta_0 \in \overline{\mathbb{D}}$, $t \mapsto \Psi_t(\zeta_0)$ is continuous in the interval $[0, 1)$. Furthermore, we may define

$$\Psi_1(\zeta) := \lim_{t\to 1^-} \Psi_t(\zeta) = \psi(\zeta, 0), \tag{4.2}$$

Thus, $\Psi_1 \in \mathcal{O}(\mathbb{D}; \mathbb{C}^k)$. Also, note that, for each $t \in [0,1]$,

$$\sup_{\zeta\in\partial\mathbb{D}} |\Psi_{t,j}(\zeta)| = \sup_{\zeta\in\partial\mathbb{D}} \left|\psi_j(\zeta, (1-t)^2\overline{\zeta})\right| < 1,\ j = 0, ..., k-1. \tag{4.3}$$

Let $\delta > 0$ be so small that

- ► $\eta \in D(0;\delta) \Rightarrow \Sigma_a + (0,\eta) \subset \Omega \cap \left(\overline{\mathbb{D}} \times \mathbb{D}\right)$; and
- ► for all $\eta \in D(0;\delta)$ and $j = 0, ..., k-1$,

$$\sup_{(\zeta,s)\in\mathcal{R}} \left|\psi_j(\zeta, s\overline{\zeta})\right| + \sum_{l=j+1}^{k-1} \binom{l}{l-j} \sup_{(\zeta,s)\in\mathcal{R}} \left|\psi_l(\zeta, s\overline{\zeta})\right| |\eta|^{l-j} + \binom{k}{k-j} |\eta|^{k-j} < 1. \tag{4.4}$$

Let $\psi^{(\eta)} = \left(\psi_1^{(\eta)}, ..., \psi_{k-1}^{(\eta)}\right) \in \mathcal{O}(\mathbb{D}^2; \mathbb{C}^k)$ be defined pointwise on $\overline{\mathbb{D}}^2$ by (2.1). By (4.4),

$$\sup_{(\zeta,s)\in\mathcal{R}} \left|\psi_j^{(\eta)}(\zeta, s\overline{\zeta})\right| < 1\ \forall\eta \in D(0;\delta) \text{ and } j = 0, ..., k-1.$$

Thus, each $\psi_j^{(\eta)} \in \mathfrak{G}_2$.

Now, just as in the proof of Theorem 1.3, we use $\{\Psi_t\}_{t\in[0,1]}$ to construct continuous families of continuous discs $\left\{\Psi_t^{(\eta)} = \left(\Psi_{t,0}^{(\eta)}, ..., \Psi_{t,k-1}^{(\eta)}\right)\right\}_{t\in[0,1]}$, on $\overline{\mathbb{D}}$, as follows:

$$\Psi_{t,j}^{(\eta)} := \Psi_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j} \binom{l}{l-j} \Psi_{t,l}\eta^{l-j} + (-1)^{k-j} \binom{k}{k-j} \eta^{k-j}. \tag{4.5}$$

Note that $\Psi_t^{(0)} = \Psi_t$, and by construction

$$\sup_{\zeta\in\partial\mathbb{D}} \left|\Psi_t^{(\eta)}(\zeta)\right| = \sup_{\zeta\in\partial\mathbb{D}} \left|\psi_t^{(\eta)}(\zeta, (1-t)^2\overline{\zeta})\right| < 1. \tag{4.6}$$

As before, fixing a domain $D \Subset \Omega$ such that $S \subset D$, we claim that the following properties are satisfied:

a*) $\Psi_0^{(\eta)} = a^{(\eta)}\ \forall\eta \in D(0;\delta)$;

b*) for a fixed t, $\Psi_t^{(\eta)}$, depends analytically on η;

c*) for each $\Psi_t^{(\eta)}$, $\Sigma_{\Psi_t^{(\eta)}} \setminus \overline{D}$ is an analytic subvariety of $\overline{\mathbb{D}} \times \mathbb{C} \setminus \overline{D}$; and

d*) for each t, $\Sigma_{\Psi_t^{(\eta)}} \subset \overline{\mathbb{D}} \times D(0;2)\ \forall\eta \in D(0;\delta)$.

Properties a*) and b*) pose no problem, and c*) can be argued in exactly the same way as in Section 3. For d^*), we write, in the notation established in Section 2:

$$\Sigma_{\Psi_t^{(\eta)},\partial Ann(0;1-t,1)} = \Sigma_{\Psi_t^{(\eta)},\partial D(0;1-t)} \bigcup \Sigma_{\Psi_t^{(\eta)},\partial\mathbb{D}}. \tag{4.7}$$

Note that $\Sigma_{\Psi_t^{(\eta)},\partial D(0;1-t)} \subset \partial D(0;1-t) \times D(0;2)$, while due to inequality (4.6) and Remark 2.3(i), we have that $\Sigma_{\Psi_t^{(\eta)},\partial\mathbb{D}} \subset \partial\mathbb{D}\times D(0;2)$. Thus, applying Lemma 2.1 (specifically, its paraphrasing in Remark 2.2) to $\Sigma_{\Psi_t^{(\eta)},Ann(0;1-t,1)}$, we have that d^*) holds.

From this, we conclude that the mapping $A : \overline{\mathbb{D}}\times[0,1] \to \mathbb{C}^k$ defined as $A(z,t) := \Psi_t(z)$ satisfies the hypotheses of Lemma 2.4. Thus, there exists a connected open neighbourhood Ω_1 of $S_1 := \Sigma_{\Psi_1^{(0)}} \cup \big(\partial\mathbb{D} \times D(0;2)\big)$ and a $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1\big|_{\Omega_1\cap\big(V\times D(0;2)\big)} \equiv f\big|_{\Omega_1\cap\big(V\times D(0;2)\big)},$$

where $V := Ann(0;1-\varepsilon,1+\varepsilon),\ \varepsilon > 0$, such that $V\times D(0;2) \subset \Omega$. $\Psi_1^{(0)}$ being holomorphic by construction, we can repeat the argument presented in the proof of Theorem 1.3 to conclude that $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega\cap\mathbb{D}^2} \equiv f\big|_{\Omega\cap\mathbb{D}^2}.$$

□

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