Some Descriptions of the Envelopes of Holomorphy of Domains in $\mathbb{C}^n$

An M.S. Thesis

Submitted, in partial fulfillment of the

requirements for the Degree of

Master of Science

in the Faculty of Science,

by

Purvi Gupta

Department of Mathematics

Indian Institute of Science

Bangalore - 560012

March, 2010

Declaration

I hereby declare that the work in this thesis has been carried out by me in the Integrated Ph.D. Program under the supervision of Dr. Gautam Bharali, and in the partial fulfillment of the requirements of the Master of Science Degree of the Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma or any other title elsewhere.

Acknowledgements

I would first like to thank my project supervisor Dr. Gautam Bharali to whom I owe much of my enthusiasm for this subject. A patient advisor, he has always encouraged me to realize my abilitites. I especially thank him for having been generous with his ideas and insights during the course of this project. I would also like to express my gratitude to the other members of the mathematics department, with whom I have interacted, for making my experience at the Indian Institute of Science enriching and memorable. Finally, a word of thanks to Rishika and Sandeep without whom coffee-breaks could not have been half as stimulating.

Abstract

It is well known that there exist domains Ω in $\mathbb{C}^n$, $n \geq 2$, such that all holomorphic functions in Ω continue analytically beyond the boundary. We wish to study this remarkable phenomenon. The first chapter seeks to motivate this theme by offering some well-known extension results on domains in $\mathbb{C}^n$ having many symmetries. One important result, in this regard, is Hartogs' theorem on the extension of functions holomorphic in a certain neighbourhood of $(\overline{\mathbb{D}} \times \{0\}) \cup (\partial\mathbb{D} \times \mathbb{D})$, $\mathbb{D}$ being the open unit disc in $\mathbb{C}$. To understand the nature of analytic continuation in greater detail, in Chapter 2, we make rigorous the notions of 'extensions' and 'envelopes of holomorphy' of a domain. For this, we use methods similar to those used in univariate complex analysis to construct the natural domains of definitions of functions like the logarithm. Further, to comprehend the geometry of these abstractly-defined extensions, we again try to deal with some explicit domains in $\mathbb{C}^n$; but this time we allow our domains to have fewer symmetries. The subject of Chapter 3 is a folk result generalizing Hartogs' theorem to the extension of functions holomorphic in a neighbourhood of $S \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$, where S is the graph of a $\overline{\mathbb{D}}$-valued function Φ, continuous in $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. This problem can be posed in higher dimensions and we give its proof in this generality. In Chapter 4, we study Chirka and Rosay's proof of Chirka's generalization (in $\mathbb{C}^2$) of the above-mentioned result. Here, Φ is merely a continuous function from $\overline{\mathbb{D}}$ to itself. Chapter 5 — a departure from our theme of Hartogs-Chirka type of configurations — is a summary of the key ideas behind a 'non-standard' proof of the so-called Hartogs phenomenon (i.e., holomorphic functions in any connected neighbourhood of the boundary of a domain $\Omega \Subset \mathbb{C}^n$, $n \geq 2$, extend to the whole of Ω). This proof, given by Merker and Porten, uses tools from Morse theory to tame the boundary geometry of Ω, hence making it possible to use analytic discs to achieve analytic continuation locally. We return to Chirka's extension theorem, but this time in higher dimensions, in Chapter 6. We see one possible generalization (by Bharali) of this result involving $\Phi \in \mathfrak{A}$, where $\mathfrak{A}$ is a subclass of $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{D}^n)$, $n \geq 2$. Finally, in Chapter 7, we consider Hartogs-Chirka type configurations involving graphs of multifunctions given by "Weierstrass pseudopolynomials". We will consider pseudopolynomials with coefficients belonging to two different subclasses of $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{C})$, and show that functions holomorphic around these new configurations extend holomorphically to $\mathbb{D}^2$.

Contents

1. Introduction and Basic Theorems

In the function theory of several complex variables, a function $f : \Omega \to \mathbb{C},\ \Omega$ a domain in $\mathbb{C}^n$, is *holomorphic* if for every point $a \in \Omega$, there is a polydisc $\triangle(a)$ centred at a such that $\triangle(a) \subset \Omega$ and f can be written as an absolutely convergent power series

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-a)^\alpha, \ \ z \in \triangle(a),$$

where the right-hand side converges uniformly on compact subsets of $\triangle(a)$. The set of functions holomorphic on Ω shall be denoted by $\mathcal{O}(\Omega)$. The two most well-known phenomena in the study of such functions are those of analytic continuation and the inequivalence of the ball and the polydisc. This is a report on an effort to study the continuation phenomenon systematically. It is easy to see that if Ω is a domain in $\mathbb{C}$ and $a \in \mathbb{C} \setminus \Omega$, there exists a holomorphic function f in Ω which cannot be continued analytically to the point a. This is not true in $\mathbb{C}^n,\ n > 1$. We begin with some well-known results to elucidate this.

Theorem 1.1 (Hartogs)**.** *Let* $\Omega := D(0; 1+\varepsilon) \times D(0;\varepsilon)^{n-1} \cup Ann(0; 1-\varepsilon, 1+\varepsilon) \times D(0;1)^{n-1}$, $n \geq 2$, *for some small* $\varepsilon > 0$*. If* $f \in \mathcal{O}(\Omega)$*, then* f *extends holomorphically to* $\mathbb{D}^n$*, i.e.,* $\exists F \in \mathcal{O}(\mathbb{D}^n)$ *such that* $F\big|_{\Omega \cap \mathbb{D}^n} \equiv f\big|_{\Omega \cap \mathbb{D}^n}$.

This theorem is a special case of a more general, yet basic, phenomenon that we now explore. For this, we need a definition.

Definition 1.2. A domian $\Omega \subseteq \mathbb{C}^n$ is called a *Reinhardt domain* if whenever $(z_1, ..., z_n) \in \Omega$ and $(\theta_1, ..., \theta_n) \in \mathbb{R}^n$, we have $(e^{i\theta_1} z_1, ..., e^{i\theta_n} z_n) \in \Omega$.

Theorem 1.3. *Let* Ω *be a Reinhardt domain in* $\mathbb{C}^n$ *and* $f \in \mathcal{O}(\Omega)$*. Then,* f *admits a Laurent series expansion*

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$$

such that the series on the right-hand side converges absolutely $\forall z \in \Omega$ *and uniformly to* f *on compact subsets of* Ω*. Moreover, the* a_α*'s are uniquely determined by* f*.*

Proof. We begin by proving the uniqueness. Let $w \in \Omega$ be a point with coordinates $(w_1, ..., w_n)$, $w_j \neq 0\ \forall j \leq n$. Let $\mathbb{T}^n(w) := \{(w_1 e^{i\theta_1}, ..., w_n e^{i\theta_n}) : (\theta_1, ..., \theta_n) \in \mathbb{R}^n\}$. Then, since the series converges uniformy to f on compact subsets of Ω and $\mathbb{T}^n(w)$ is compact in Ω, we can multiply by $e^{-i(\alpha_1\theta_1+\cdots+\alpha_n\theta_n)}$ and integrate term by term to obtain:

$$a_\alpha = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{f(w_1e^{i\theta_1},...,w_ne^{i\theta_n})e^{-i(\alpha_1\theta_1+\cdots+\alpha_n\theta_n)}}{w^\alpha}d\theta_n...d\theta_1. \tag{1.1}$$

As clarification: we shall use multi-index notation throughout this chapter. In this notation, for any $\alpha \in \mathbb{Z}^n$ and $w \in \mathbb{C}^n$, we write $w^\alpha := w_1^{\alpha_1}\dots w_n^{\alpha_n}$. The above expression holds for any Laurent series expansion having the desired convergence. Note that it does not depend on w. Hence, a_α's are uniquely determined by f.

To prove the existence of an expansion as above, we first note that if

$$\mathcal{A} := \{z \in \mathbb{C}^n | r_j < |z_j| < R_j,\ 0 \le r_j < R_j,\ j = 1, ..., n\}$$

and f is holomorphic on $\mathcal{A}$, then, by iteration of the Laurent expansion for functions of one complex variable defined on annuli, it follows that f has an expansion in a Laurent series. Let $w \in \Omega$. Let $\varepsilon > 0$ be so small that Ω, being Reinhardt, contains the set

$$\mathcal{A}(w;\varepsilon) := \{z \in \mathbb{C}^n : |w_j| - \varepsilon < |z_j| < |w_j| + \varepsilon\}.$$

Since this is a set of the form $\mathcal{A}$ above, there exists a Laurent series expansion

$$\sum_{\alpha\in\mathbb{Z}^n} a_\alpha(w)z^\alpha = f(z)\ ,\ z \in \mathcal{A}(w;\varepsilon),$$

which converges uniformly to f on compact subsets of $\mathcal{A}(w;\varepsilon)$. Now, if $\widetilde{w} \in \mathcal{A}(w;\varepsilon)$ and $\sum a_\alpha(\widetilde{w})z^\alpha$ is the expansion corresponding to $\widetilde{w}$ in a set $\mathcal{A}(\widetilde{w};\varepsilon) \subseteq \Omega$, then the uniqueness assertion above shows that $a_\alpha(w) = a_\alpha(\widetilde{w})$.

Hence, the function $w \mapsto a_\alpha(w)$ is locally constant on Ω for any $\alpha \in \mathbb{Z}^n$. Since Ω is connected, $a_\alpha(w) = a_\alpha$ is independent of w. This establishes the existence of a Laurent series expansion

$$\sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha = f(z)$$

that converges absolutely at each $z \in \Omega$. Now, let K be a compact set in Ω. Then, there exist $w_1, ..., w_m$ and $\varepsilon_1, ..., \varepsilon_m$ such that $K \subseteq \bigcup_{s=1}^{m} \mathcal{A}(w_s;\varepsilon_s)$. Since the series converges uniformly on compacts in each $\mathcal{A}(w_s;\varepsilon_s)$, we obtain uniform convergence in K. □

The above theorem (Theorem 1.3) is often useful in proving results regarding analytic continuation of holomorphic functions beyond a given domain. Here is one such result.

Corollary 1.4. *Let Ω be a Reinhardt domain such that for each j, $1 \le j \le n$, there is a point $z \in \Omega$ whose j-th coordinate is 0. If $f \in \mathcal{O}(\Omega)$, then $\exists F \in \mathcal{O}(\widetilde{\Omega})$, where $\widetilde{\Omega} =$*

$\{(\rho_1 z_1, ..., \rho_n z_n) \big| \; 0 \leq \rho_j \leq 1, (z_1, ..., z_n) \in \Omega\}$, *such that* $F\big|_\Omega \equiv f$.

Proof. Let $f \in \mathcal{O}(\Omega)$. Then, by Theorem 1.3, there exists a Laurent series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$$

such that the series on the right-hand side converges absolutely $\forall z \in \Omega$ and uniformly to f on compact subsets of Ω. Let $\eta : \mathbb{N} \to \mathbb{Z}^n$ be an enumeration. Then,

$$\sum_{j=0}^{\gamma} a_{\eta(j)} z^{\eta(j)} \longrightarrow f(z) \text{ absolutely } \forall z \in \Omega. \tag{1.2}$$

Now, suppose there exists an $\alpha_0 = (\alpha_0^1, ..., \alpha_0^n)$ and a $k \leq n$ such that

- $\alpha_0^k < 0$; and
- $a_{\alpha_0} \neq 0$.

By the hypothesis imposed on Ω, there exists a $z_0 \in \Omega$ such that $z_k = 0$ and $z_j \neq 0$, when $j \neq k$. If $p \in \mathbb{N}$ is such that $\eta(p) = \alpha_0$, then

$$\sum_{j=0}^{\gamma} |\alpha_{\eta(j)} z^{\eta(j)}| = \infty \quad \forall \gamma \geq p.$$

This contradicts equation (1.2). Hence, $a_{\alpha_0} = 0$. This implies that there is, in fact, a power series expansion of f in Ω as follows:

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha. \tag{1.3}$$

Finally, define $F : \widetilde{\Omega} \to \mathbb{C}$ as

$$F(z) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \tag{1.4}$$

where the absolute convergence of the series on the right-hand side, for each $z \in \widetilde{\Omega}$, follows from the defintion of $\widetilde{\Omega}$. Uniform convergence on compact sets of the type $\mathcal{D} := \overline{D(z_1; \varepsilon)} \times \cdots \times \overline{D(z_n; \varepsilon)}$, $\mathcal{D}$ contained in $\widetilde{\Omega}$, is now established by dominating $\big| \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \big|$ by $\sum_{\alpha \in \mathbb{N}^n} |a_\alpha| \varepsilon^{\alpha_1 + \cdots + \alpha_n}$ for z in $\mathcal{D}$. Since any compact subset of $\widetilde{\Omega}$ can be covered by a finite number of such sets lying within Ω, the convergence of the right-hand side of equation (1.4) is uniform on compact subsets of $\widetilde{\Omega}$. Thus $F \in \mathcal{O}(\widetilde{\Omega})$ and, by equation (1.3), $F\big|_\Omega \equiv f$. □

Observe that Theorem 1.1 is, as hinted earlier, a general case of Corollary 1.4 as the Ω defined there is a Reinhardt domain containing the origin.

We will revisit the method of proof adopted above in Chapter 3. The following result is proved using somewhat different techniques.

Theorem 1.5 (Hartogs, [9])**.** *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$. Let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. For each $f \in \mathcal{O}(\Omega \setminus K)$, $\exists \widetilde{f} \in \mathcal{O}(\Omega)$ such that $\widetilde{f}\big|_{\Omega\setminus K} \equiv f$.*

The key to this theorem lies in the following proposition:

Proposition 1.6. *Let $f = \sum_{j=1}^{n} f_j d\bar{z}_j$ be a $(0,1)$ form on $\mathbb{C}^n$, $n \geq 2$, with $\mathcal{C}^k_c$-smooth coefficients, that is $\bar{\partial}$-closed. Then the equation*

$$\bar{\partial} u = f \tag{1.5}$$

has a $\mathcal{C}^k$-smooth solution u such that u is compactly supported; indeed, $u \equiv 0$ on the unbounded component of $(supp(f))^{\complement}$.

Equation (1.5) is a special case of a class of partial differential equations which play an important role in analysing the continuation phenomenon. Finding conditions under which such equations, i.e., $\bar{\partial} u = f$, where f can be subjected to various hypotheses, admit a solution is called the $\bar{\partial}$-problem. Often, the question of extending holomorphic functions beyond a given domain can be reduced to establishing the solvability of a specific $\bar{\partial}$-equation. The proof of Theorem 1.5 given below is one such instance. We will see another example in Chapter 4.

Proof of Theorem 1.5. Let $f \in \mathcal{O}(\Omega \setminus K)$. We can find a set $U \Subset \Omega$ such that U is open and $K \subset U \Subset \Omega$. Let V be an open set in U such that $K \subset V \Subset U$. Now, let $\chi \in \mathcal{C}^\infty_c(\mathbb{C}^n)$ be such that

$$\chi(z) = \begin{cases} 1, & \text{if } z \in \overline{V}, \\ 0, & \text{if } z \in U^{\complement}. \end{cases}$$

Define $F : \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} (1-\chi)(z) f(z), & \text{if } z \in \Omega \setminus V, \\ 0, & \text{if } z \in \overline{V}. \end{cases}$$

As $\partial V \subset \Omega \setminus K$, $F \in \mathcal{C}^\infty(\Omega)$. Also, $F|_{\Omega\setminus\overline{U}} \equiv f$.

Now, if we could obtain a correction term, say u, on Ω such that $\widetilde{f} := F - u$ is holomoprhic on Ω and satisfies $\widetilde{f}\Big|_N \equiv F\Big|_N$, where N is some open subset of $\Omega \setminus \overline{U}$, then, since $\Omega \setminus K \supset \Omega \setminus \overline{U}$ is connected and $F|_{\Omega\setminus\overline{U}} \equiv f$, $\widetilde{f}$, by the Identity Theorem, would be the required extension. For this purpose, let

$$\phi_j(z) := \frac{\partial F}{\partial \bar{z}_j}(z) = \begin{cases} -f(z)\dfrac{\partial \chi}{\partial \bar{z}_j}(z), & \text{if } z \in \Omega \setminus V, \\ 0, & \text{if } z \in \overline{V} \end{cases}$$

and consider the $\bar{\partial}$-problem

$$\frac{\partial u}{\partial \bar{z}_j} = \widetilde{\phi}_j,$$

where

$$\widetilde{\phi}_j(z) = \begin{cases} \phi_j(z), & \text{if } z \in \Omega, \\ 0, & \text{if } z \in \mathbb{C}^n \setminus \Omega, \end{cases}$$

$\forall j = 1, ..., n$. As $F \in \mathcal{C}^\infty(\Omega)$ and $\phi_j|_{\Omega \setminus \overline{U}} \equiv 0$, $\widetilde{\phi}_j \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ $\forall j = 1, ...n$ and $\dfrac{\partial \widetilde{\phi}_k}{\partial \bar{z}_j} = \dfrac{\partial \widetilde{\phi}_j}{\partial \bar{z}_k}$ $\forall j, k \leq n$. Hence, by Proposition 1.6, there exists a $\widetilde{u} \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that

$$\frac{\partial \widetilde{u}}{\partial \bar{z}_j} = \widetilde{\phi}_j.$$

Let $u := \widetilde{u}\big|_{\Omega}$. If $\widetilde{f} := F - u$, then $\widetilde{f} \in \mathcal{O}(\Omega)$. Also, since $\text{supp}(\widetilde{\phi}_j) \subseteq \overline{U}$, $j \leq n$, $\widetilde{u}$ vanishes on the unbounded component of $\overline{U}^{\mathsf{C}}$. But there exists an open set N in the unbounded component of $\overline{U}^{\mathsf{C}}$ such that $N \subset \Omega \setminus \overline{U}$. Hence, $u\big|_N \equiv 0$ and we have, $\widetilde{f}\Big|_N \equiv F\big|_N \equiv f\big|_N$. As observed earlier, the connectedness of $\Omega \setminus K$ implies that $\widetilde{f}$ is the required extension. $\square$

We conclude this chapter with a folk result. This result is often quoted, but we were unable to find a proof in the literature. In the course of finding a proof, we were able to prove a somewhat more general result. This will be discussed in Chapter 3.

Theorem 1.7. *Let $\Phi \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ be such that $\Phi(\partial\mathbb{D}) \subseteq \overline{\mathbb{D}}$. Let Ω be a connected neighbourhood of $S :=$ graph (Φ) $\cup$ $(\partial\mathbb{D}) \times \overline{\mathbb{D}}$ such that $\Omega \cap \mathbb{D}^2$ is connected. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^2$.*

Observe that if we let Φ be the constant function 0, then we obtain Theorem 1.1 as a particular case of the above theorem.

2. Envelopes of Holomorphy

Given the results of the previous chapter, one is now inspired to ask whether, given a domain $\Omega \subset \mathbb{C}^n$, a maximal domain can, in some meaningful manner, be produced such that all functions $f \in \mathcal{O}(\Omega)$ simultaneously extend to it. Such a domain, if it exists, is called an envelope of holomorphy of Ω. In view of what we know about the maximal domain of existence of the logarithm in univariate complex analysis, an envelope of holomorphy is not, in general, expected to be a domain in $\mathbb{C}^n$. This motivates the following definition.

Definition 2.1. A *Riemann domain over* $\mathbb{C}^n$ is a pair (Ω, p), where Ω is a topological space, and $p : \Omega \to \mathbb{C}^n$ is a local homeomorphism.

In this case, a continuous function $f : \Omega \to \mathbb{C}$ is called *holomorphic (relative to p)* if, for every $a \in \Omega$, there is a neighbourhood $U \ni a$ such that

- $p\big|_U$ is a homeomorphism onto $p(U) \subset \mathbb{C}^n$; and
- the function $f \circ (p\big|_U)^{-1}$ is holomorphic on $p(U)$.

If (Ω', p') is a Riemann domain over $\mathbb{C}^m$, a continuous map $u : \Omega \to \Omega'$ is called holomorphic if, for any open set $V' \subset \Omega'$ and f' holomorphic on V', the function $f' \circ u$ is holomorphic on $u^{-1}(V')$. If, in addition, u is a homeomorphism of Ω onto Ω' and the inverse is also holomorphic, then we say that u is an *isomorphism*. As before, the set of holomorphic functions on Ω is denoted by $\mathcal{O}(\Omega)$. We will often use the fact that the Identity Theorem holds for Riemann domains over $\mathbb{C}^n$ as well.

Now, if $f \in \mathcal{O}(\Omega)$, what does it mean to say that f can be continued analytically to another Riemann domain over $\mathbb{C}^n$? This will be clear after our next definition.

Definition 2.2. Let (Ω, p_0) be a connected Riemann domain over $\mathbb{C}^n$ and $S \subset \mathcal{O}(\Omega)$. We say that $\{(X, p); \phi : \Omega \to X\}$, where (X, p) is a connected Riemann domain over $\mathbb{C}^n$ and $\phi : \Omega \to X$ is a continuous map such that $p \circ \phi = p_0$, is an S-*extension* of (Ω, p_0) if, to every $f \in S$, there is an $F_f \in \mathcal{O}(X)$ such that $F_f \circ \phi = f$.

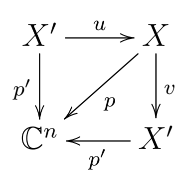
Remark. F_f is uniquely determined for each $f \in S$. (First on $\phi(\Omega)$, since $F_f \circ \phi = f$, hence on X by the Identity Theorem). It is called the *extension* of f to X.

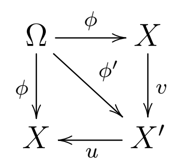
The notion of an S-extension of holomorphy being the maximal domain of analytic continuation of each $f \in S$ is captured by the following definition.

Definition 2.3. Let (Ω, p_0) be a connected Riemann domain over $\mathbb{C}^n$ and $S \subset \mathcal{O}(\Omega)$. An *$S$-envelope of holomorphy* of (Ω, p_0) is an S-extension $\{(X, p); \phi : \Omega \to X\}$, such that the following holds:
$(*)$ For any S-extension $\{(X', p'); \phi' : \Omega \to X'\}$ of (Ω, p_0), there is a holomorphic map $u : X' \to X$ such that $p' = p \circ u$, $\phi = u \circ \phi'$ and $F'_f = F_f \circ u$ for all $f \in S$, where F_f and F'_f are the extensions of $f \in S$ to X and X' respectively. An *envelope of holomorphy* of (Ω, p_0) is simply an S-envelope of holomorphy with $S = \mathcal{O}(\Omega)$.

Remarks. (i) Note that u in $(*)$ is unique since it is determined on $\phi'(\Omega)$ by the equation $u \circ \phi' = \phi$.

(ii) The S-envelope of holomorphy of (Ω, p_0), if it exists, is unique up to isomorphism. In fact, let $\{(X, p); \phi : \Omega \to X\}$ and $\{(X', p'); \phi' : \Omega \to X'\}$ be two S-envelopes of holomprhy of (Ω, p_0). Then, by $(*)$ of Definition 2.3, there are holomorphic maps $u : X' \to X$ and $v : X \to X'$ such that $p = p' \circ v$, $p' = p \circ u$, $\phi = u \circ \phi'$ and $\phi' = v \circ \phi$, i.e. the following two diagrams commute:





Then, $u \circ v \circ \phi = u \circ \phi' = \phi$, so that $u \circ v$ is the identity on $\phi(\Omega)$. Similarly, $v \circ u \equiv$ identity on $\phi'(\Omega)$. Hence by the Identity Theorem, u is an isomorphism of X' onto X.

Having made rigorous the concept of the envelope of holomorphy of a Riemann domain, we now proceed to examine whether there exist any Riemann domains for which such maximal extensions exist. The following theorem establishes the existence of the envelope of holomorphy for every Riemann domain over $\mathbb{C}^n$ by an explicit construction.

Theorem 2.4 (Cartan-Thullen, [3])**.** *Let (Ω, p_0) be a connected Riemann domain over $\mathbb{C}^n$ and $S \subset \mathcal{O}(\Omega)$. The S-envelope of holomorphy of (Ω, p_0) exists.*

For the purpose of proving the above, we first introduce the sheaf of S-germs of holomorphic functions on $\mathbb{C}^n$. Let S be a set and $a \in \mathbb{C}^n$. Set

$$\mathfrak{S}^a := \{(U, \{f_s\}_{s\in S}) | U \text{ is an open set containing } a \text{ and each } f_s \text{ is holomorphic on } U.\}$$

We say that two elements of $\mathfrak{S}^a$, say $(U, \{f_s\}_{s\in S})$ and $(V, \{g_s\}_{s\in S})$, are equivalent if there exists a neighbourhood W of a, $W \subset U \cap V$, such that, for all $s \in S$, $f_s\big|_W \equiv g_s\big|_W$. An equivalence class with respect to this relation is called an *S-germ of holomorphic functions* at a. The set of such S-germs is denoted by $\mathcal{O}^a_S$. The set $\mathcal{O}_S := \bigcup_{a\in\mathbb{C}^n} \mathcal{O}^a_S$ is called the *sheaf of S-germs of holomorphic functions* on $\mathbb{C}^n$. There is a natural projection $p = p_S : \mathcal{O}_S \to \mathbb{C}^n$ defined by $p(\underline{f}) = a$ when $\underline{f} \in \mathcal{O}^a_S$.

Now, a topology on $\mathcal{O}_S$ is defined as follows: Let $\underline{f_a} \in \mathcal{O}_S^a$ and $(U, \{f_s\}_{s\in S})$ be a representative of $\underline{f_a}$. Let $\underline{f_b}$ be the S-germ defined by $\{f_s\}_{s\in S}$ at $b \in U$, and let $N(U, \{f_s\}_{s\in S}) := \{\underline{f_b} | b \in U\}$. The collection of sets

$$\mathcal{N}^{\underline{f_a}} := \left\{ N(U, \{f_s\}_{s\in S}) : (U, \{f_s\}_{s\in S}) \text{ is a representative of } \underline{f_a} \right\},$$

forms a fundamental system of neighbourhoods of $\underline{f_a}$. It turns out that $(\mathcal{O}_S, p_S)$ is a Riemann domain over $\mathbb{C}^n$.

Proof of Theorem 2.4. For any (Ω, p_0) and $S \subset \mathcal{O}(\Omega)$, we define a map $\phi = \phi(p_o, S)$ from Ω into $\mathcal{O}_S$ as follows. Let $a \in \Omega$ and $a_0 = p_0(a) \in \mathbb{C}^n$. Let U be an open neighbourhood of a such that $p_0\big|_U$ is an isomorphism onto an open set $U_0 \subset \mathbb{C}^n$. Let $\underline{g_a}$ be the S-germ at a_0 defined by the pair $(U_0, \{f_s\}_{s\in S})$, where $f_s = s \circ (p_0\big|_U)^{-1}, s \in S$. We set $\phi(a) = \underline{g_a}$. That ϕ is continuous and $p \circ \phi = p_0$,where $p : \mathcal{O}_S \to \mathbb{C}^n$ is the natural projection, is easily verified by examining the definition of ϕ and the topology imposed on $\mathcal{O}_S$. Furthermore, p and ϕ are local homeomorphisms. In view of the relation $p \circ \phi = p_0$, ϕ is in fact a local isomorphism.

Since Ω is connected, so is $\phi(\Omega)$. Let X be the connected component of $\mathcal{O}_S$ containing $\phi(\Omega)$, and denote again by p the restriction to X of the map $p : \mathcal{O}_S \to \mathbb{C}^n$. We claim that $\{(X, p); \phi : \Omega \to X\}$ is an S-envelope of holomorphy of Ω.

To see this, we first observe that, for all $s \in S$, we have a holomorphic function F_s on $\mathcal{O}_S$ defined as follows. If $\underline{g_z} \in \mathcal{O}_S^z$ is defined by $(V, \{g_s\}_{s\in S})$, we set $F_s(\underline{g_z}) = g_s(z)$. The holomorphicity of F_s is immediate from the definition of the natural projection p from $\mathcal{O}_S$ to $\mathbb{C}^n$. We denote the restriction of F_s to X again by F_s. Now, by the very definition of ϕ, it follows that $F_s \circ \phi = s$ for all $s \in S$. Hence, $\{(X, p); \phi : \Omega \to X\}$ is an S-extension of (Ω, p_0).

To prove that it is, indeed, the S-envelope of holomorphy of (Ω, p_0), let $\{(X', p'); \phi' : \Omega \to X'\}$ be given with $p' \circ \phi' = p_0$ and suppose that for all $s \in S$, there exists $F'_s \in \mathcal{O}(X')$ such that $s = F'_s \circ \phi'$. Let $S' = \{F'_s\}_{s\in S}$ and $u : X' \to \mathcal{O}_S$ be the map $\phi(p', S')$ (defined in the beginning of the proof). Since $F'_s \circ \phi' = s$ and $p' \circ \phi' = p_0$, we have $\phi = u \circ \phi'$ (locally, $F'_s \circ p^{-1} = F'_s \circ \phi' \circ \phi'^{-1} \circ p'^{-1} = s \circ p_0^{-1}$). Clearly, $p' = p \circ u$. □

Before moving ahead, we must attempt to demystify the above construction. What insight lies behind realising the envelope of holomorphy of a Riemann domain as a certain path component of the sheaf of S-germs of holomorphic functions? While exploring the phenomenon of analytic continuation in univariate complex analysis, one exploits the concept of analytic continuation along paths via chains of discs. However, this procedure can lead to multi-valuedness around the initial point when one analytically continues a germ of analytic function along a closed path. Therefore, we are led to consider the collection of all possible germs of holomorphic functions — i.e. the sheaf of germs of holomorphic functions over $\mathbb{C}$. Now, given a germ $\underline{f_a}$ and a path γ starting at a along which this germ can be analytically continued, we resolve the problem of multi-valuedness by considering a lifting of γ to the sheaf of germs of holomorphic functions and viewing the analytic continuation of (U, f) – i.e. a representative of $\underline{f_a}$ – as a true function defined on a suitable subset of the sheaf of germs of holomorphic functions. It is this well-known construction that motivates the choice of $\mathcal{O}_S$, or rather, a suitable connected component of it, as the S-envelope of holomorphy of a given Riemann domain. In our case, since we try to extend more than one function at the same time, we are led to work with S-germs.

Proposition 2.5. *Let (Ω, p_0) be a connected Riemann domain over $\mathbb{C}^n$ and $f \in \mathcal{O}(\Omega)$. Let F be its extension to the envelope of holomorphy (X, p). Then, $f(\Omega) = F(\Omega)$. In particular, if f is bounded, $|f(x)| < M$ for all $x \in \Omega$, then F is bounded and $|F(x)| < M$ for all $x \in X$.*

Proof. Since $f = F \circ \phi$, we have $f(\Omega) \subset F(X)$. Suppose that there exists a $c \in F(X) \setminus f(\Omega)$. Then, $\frac{1}{(f-c)} \in \mathcal{O}(\Omega)$. If G is its extension to X, then $G \cdot (F - c)$ is the extension to X of $1 = (f - c)^{-1} \cdot (f - c)$, so that $G \cdot (F - c) \equiv 1$ on X. This implies that $F(x) \neq c$ for all $x \in X$, a contradiction. □

Looking back at our intuitive notion of an envelope of holomorphy, we expect to achieve nothing new by constructing its envelope of holomorphy. To realise this in terms of Riemann domains, we first make a definition.

Definition 2.6. Let (Ω, p_0) be a connected Riemann domain over $\mathbb{C}^n$ and $S \subset \mathcal{O}(\Omega)$. Ω is called an *S-domain of holomorphy* if the natural map of Ω into its S-envelope of holomorphy is an isomorphism. If $S = \mathcal{O}(\Omega)$, Ω is simply called a *domain of holomorphy.*

That the envelope of holomorphy of a Riemann domain over $\mathbb{C}^n$ is a domain of holomorphy, is a consequence of the following proposition. The proof of this proposition suggests why we might be interested in studying S-envelopes of holomorphy when $S \subsetneq \mathcal{O}(\Omega)$.

Proposition 2.7. *Let (Ω, p_0) and (Ω', p_0') be connected Riemann domains over $\mathbb{C}^n$, and $\{(X, p); \phi : \Omega \to X\}$ and $\{(X', p'); \phi' : \Omega' \to X'\}$ their envelopes of holomorphy. Let $u : \Omega \to \Omega'$ be a holomorphic map which is a local isomorphism. Then, there exists a holomorphic map $\widetilde{u} : X \to X'$ such that the diagram*

$$\begin{array}{ccc} \Omega & \xrightarrow{u} & \Omega' \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi'} \\ X & \xrightarrow{\widetilde{u}} & X' \end{array}$$

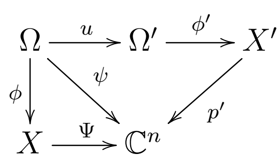
commutes.

We use the following results to prove the above proposition. Their proofs are elementary, and we shall skip them.

Lemma 2.8. *Let* (Ω, p_0) *be a Riemann domain over* $\mathbb{C}^n$ *and* $f : \Omega \to \mathbb{C}^n$ *be a holomorphic map. Suppose that* $det(df)_a \neq 0$ *for some* $a \in \Omega$*. Then there exist neighbourhoods* U *of* a *and* V *of* $f(a)$ *such that* $f(U) \subset V$ *and* $f\big|_U$ *is an isomorphism into* V*.*

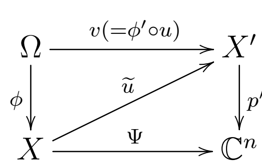
Lemma 2.9. *Let* (Ω, p_0) *and* (Ω', p_0') *be connected Riemann domains over* $\mathbb{C}^n$*, and* $\{(X', p'); \phi' : \Omega' \to X'\}$ *be the* T*-envelope of holomorphy of* (Ω', p_0')*,* $T \subset \mathcal{O}(\Omega')$*. Let* $u : \Omega \to \Omega'$ *be a local holomorphic isomorphism, and let* $S = \{f \circ u | f \in T\}$*. Then,* $\{(X', p'); \phi' \circ u : \Omega \to X'\}$ *is the* S*-envelope of holomorphy of* $(\Omega, p_0' \circ u)$*.*

Proof of Proposition 2.7. Let $v = \phi' \circ u$. Since, u and, as was shown in the proof of Theorem 2.4, ϕ' are local holomorphic isomorphisms, so is v. Consider the map $\psi = p' \circ v : \Omega \to \mathbb{C}^n$. The $\psi = (\psi_1, ..., \psi_n)$ is also a local isomorphism and each ψ_j is holomorphic on Ω (relative to p_0). Let η be the *complex* Jacobian determinant $\eta = det\left(\dfrac{\partial \psi_i}{\partial x_j}\right)$. Then, since ψ is a local isomorphism, $\eta(x) \neq 0$ for all $x \in \Omega$. Let Ψ_j be the extension of ψ_j to X, and let $\Psi = (\Psi_1, ..., \Psi_n)$. Note that Ψ is such that the following diagram commutes:



Let H be the extension of η to X. Then, by the Identity Theorem, $H = det\left(\dfrac{\partial \Psi_i}{\partial x_j}\right)$. By Proposition 2.5, $H(x) \neq 0$ for all $x \in X$. Hence, by Lemma 1, $\Psi : X \to \mathbb{C}^n$ is a local isomorphism. Moreover $\Psi \circ \phi = \psi (= p' \circ \phi' \circ u)$.

Now, consider (Ω, ψ) and $\{(X', p'); \phi' \circ u = v : \Omega \to X'\}$. Let $S = \{f \circ u | f \in \mathcal{O}(\Omega')\} = \{F \circ v | F \in \mathcal{O}(X')\}$. Since, $S \subset \mathcal{O}(\Omega)$ (relative to ψ), by Lemma 2, $\{(X', p'); v : \Omega \to X'\}$ is the S-envelope of holomorphy of (Ω, ψ). Now any holomorphic function on Ω can be extended to X, so that $\{(X, \Psi); \phi : \Omega \to X\}$ is an S-extension of (Ω, ψ). Since (X', p') is the S-envelope of holomorphy of (Ω, ψ), there is a holomorphic map $\widetilde{u} : X \to X'$ such that $p' \circ \widetilde{u} = \Psi$ and $\widetilde{u} \circ \phi = v$,i.e. the following diagram commutes:



□

This immediately leads us to our desired goal.

Corollary 2.10. *If* $\{(X, p); \phi : \Omega \to X\}$ *is the envelope of holomorphy of a Riemann domain over* $\mathbb{C}^n$ (Ω, p_0)*, then* (X, p) *is a domain of holomorphy.*

Proof. Let $\{(X', p'); \phi' : X \to X'\}$ be the envelope of holomorphy of (X, p). We see from the proof of theorem 2.4 that $\phi : \Omega \to X$ is a local analytic isomorphism. Hence, by Lemma 2, $\{(X', p'); \phi' \circ \phi : \Omega \to X'\}$ is the S-envelope of holomorphy of $(\Omega, p \circ \phi)$, where $S = \{F \circ \phi : F \in \mathcal{O}(X)\}$. But $S = \mathcal{O}(\Omega)$. Hence, $\{(X', p'); \phi' \circ \phi : \Omega \to X'\}$ is the envelope of holomorphy of $(\Omega, p \circ \phi)$. Consider the identity map from $(\Omega, p \circ \phi)$ onto (Ω, p_0). Applying Proposition 2.7, we get a map $\widetilde{\phi} : X' \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} \Omega & \xrightarrow{id} & \Omega \\ {\scriptstyle \phi'\circ\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X' & \xrightarrow{\widetilde{\phi}} & X \end{array}$$

Therefore, $\widetilde{\phi} \circ \phi' \circ \phi = \phi$. This implies that $\widetilde{\phi} \circ \phi'\big|_{\phi(\Omega)} \equiv id\big|_{\phi(\Omega)}$ and, therefore, $\widetilde{\phi} \circ \phi'$ is the identity on X. Similarly, since $\phi' \circ \widetilde{\phi} \circ \phi' \circ \phi = \phi' \circ \phi$, $\phi' \circ \widetilde{\phi}$ is identity on X'. Hence, ϕ' is an isomorphism. $\square$

We close this chapter by comparing all of the above concepts with the classical definition of a domain of holomorphy for a domain $\Omega \subsetneq \mathbb{C}^n$ (note that this *is* a Riemann domain (Ω, p_0) over $\mathbb{C}^n$, where p_0 is just the inclusion map). In view of the sheaf-theoretic construction of of the envelope of holomorphy and Corollary 2.10, $\Omega \subsetneq \mathbb{C}^n$ is *not* a domain of holomorphy precisely if there is a path $\gamma : [0, 1] \to \mathbb{C}^n$ such that:

- $\gamma(0) \in \Omega$,
- The set $\gamma([0, 1])$ crosses the boundary of Ω, and
- the $\mathcal{O}(\Omega)$-germ at $\gamma(0)$ represented by $(\Omega, \mathcal{O}(\Omega))$ can be analytically continued along γ.

The following definition — which we encounter when working with domains in $\mathbb{C}^n$ — is just a paraphrasing of the above without reference to any paths γ.

Definition 2.11. A domain $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there does *not* exist any pair (Ω_1, Ω_2) of open sets that satisfy the following:

(i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$,

(ii) Ω_2 is connected and $\Omega \subsetneq \Omega \cup \Omega_2$; and

(iii) For each $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\Omega_2)$ such that $f\big|_{\Omega_1} \equiv F\big|_{\Omega_1}$.

3. A Folk Theorem

In Chapter 2, we saw an abstract construction of the envelope of holomorphy of a domain $\Omega \subseteq \mathbb{C}^n,\ n \geq 2$ (in fact, we saw this for a Riemann domain spread over $\mathbb{C}^n$), but it is very difficult to infer its geometry. Theorem 1.1 shows that the envelope of holomorphy of the Ω discussed there is $\mathbb{D}^n$. This result is proved by exploiting the symmetries possessed by Ω (Ω is Reinhardt). The aim of this and the next chapter is to deduce the geometry of the envelope of holomorphy of Ω when it has fewer symmetries.

We start with the following theorem which is, as mentioned in Chapter 1, a generalization of a folk result (Theorem 1.7). Note that the conclusion of this theorem is same as that of Theorem 1.1, but Ω is decidedly not Reinhardt in general.

Theorem 3.1. *Let $\Phi \in \mathcal{O}(\mathbb{D}^k; \mathbb{C}^{n-k}) \cap \mathcal{C}(\overline{\mathbb{D}}^k; \mathbb{C}^{n-k})$ be such that $\Phi(\partial(\mathbb{D}^k)) \subseteq \overline{\mathbb{D}}^{n-k}$. Let Ω be a connected neighbourhood of $S :=$ graph $(\Phi) \cup (S^1)^k \times \overline{\mathbb{D}}^{n-k}$ such that $\Omega \cap \mathbb{D}^n$ is connected. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^n$.*

Proof. Let $f \in \mathcal{O}(\Omega)$. Since $(S^1)^k \times \overline{\mathbb{D}}^{n-k}$ is compact and Ω is an open set containing $(S^1)^k \times \overline{\mathbb{D}}^{n-k}$, $\exists \varepsilon > 0$ such that $Ann(0; 1-\varepsilon, 1+\varepsilon)^k \times D(0; 1+\varepsilon)^{n-k} \subseteq \Omega$. So, for each fixed $w \in D(0; 1+\varepsilon)^{n-k}$, we can define

$$f_w(z) = f(z, w)$$

which is well-defined and holomorphic in $Ann(0; 1-\varepsilon, 1+\varepsilon)^k$. As $Ann(0; 1-\varepsilon, 1+\varepsilon)^k$ is a Reinhardt domain, each f_w, by Theorem 1.3, has a Laurent series expansion as follows:

$$\left(f|_{Ann(0;1-\varepsilon,1+\varepsilon)^k \times D(0;1+\varepsilon)^{n-k}}\right)(z, w) = f_w(z) = \sum_{\alpha \in \mathbb{Z}^k} a_\alpha(w) z^\alpha$$

where

$$\begin{aligned} a_\alpha(w) &= \frac{1}{(2\pi i)^k} \int_{|z_1|=1} \cdots \int_{|z_k|=1} \frac{f_w(z_1, z_2, ..., z_k)}{z_1^{\alpha_1+1} \cdots z_k^{\alpha_k+1}} dz_k ... dz_1 \\ &= \frac{1}{(2\pi i)^k} \int_{|z_1|=1} \cdots \int_{|z_k|=1} \frac{f(z_1, z_2, ..., z_k, w)}{z_1^{\alpha_1+1} \cdots z_k^{\alpha_k+1}} dz_k ... dz_1, \end{aligned} \tag{3.1}$$

for all $\alpha \in \mathbb{Z}^k$ and $w \in D(0; 1+\varepsilon)^{n-k}$. Using Leibniz's theorem for differentiating under the integral sign, we observe that $a_\alpha(w) \in \mathcal{O}(D(0; 1+\varepsilon)^{n-k})\ \forall \alpha \in \mathbb{Z}^k$.

Since graph$(\Phi) \subseteq \Omega$ is compact, $\exists \delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and $\Delta((z, \Phi(z)); (2\delta, ..., 2\delta)) \subseteq \Omega$ for every $z \in \overline{\mathbb{D}}^k$. Fix $w \in D(0;\delta)^{n-k}$. Now, for any $\zeta \in D(0; 1+\delta) \subseteq \mathbb{C}$ and $z \in (S^1)^k \subseteq \mathbb{C}^k$, $|(\zeta\Phi(z) + w)_j| < 1 + 2\delta < 1 + \varepsilon$; $j = 1, ..., n-k$, where z_j represents the j^{th} co-ordinate of $z \in \mathbb{C}^{n-k}$. So, we can define

$$G_\alpha^w(\zeta) = \frac{1}{(2\pi i)^k} \int_{|z_1|=1} \cdots \int_{|z_k|=1} \frac{f(z_1, z_2, ..., z_k, \zeta\Phi(z_1, z_2, ..., z_k) + w)}{z_1^{\alpha_1+1} \cdots z_k{}^{\alpha_k+1}} dz_k...dz_1$$

for ζ in $D(0; 1+\delta)$. Note that, for a fixed $w \in D(0;\delta)^{n-k}$, we may differentiate under the integral sign to obtain:

$$\begin{aligned} &\left.\frac{\partial G_\alpha^w}{\partial \bar{\zeta}}\right|_\zeta \\ =&\frac{1}{(2\pi i)^k} \int_{\mathbb{T}^k} \frac{\sum_{l=1}^{n-k} \left[\frac{\partial f}{\partial z_{l+k}}(z, \zeta\Phi(z) + w)\ \partial_{\bar{\zeta}}(\zeta\Phi(z) + w)_l\Big|_\zeta + \frac{\partial f}{\partial \bar{z_{l+k}}}(z, \zeta\Phi(z) + w)\ \partial_{\bar{\zeta}}(\overline{\zeta\Phi(z) + w})_l\Big|_\zeta \right]}{z_1^{\alpha_1+1} z_2^{\alpha_2+1} \cdots z_k{}^{\alpha_k+1}} dz \\ =&0 \ \ \forall \zeta \in D(0; 1+\delta). \end{aligned}$$

Hence, $G_\alpha^w \in \mathcal{O}(D(0; 1+\delta))$ $\forall \alpha \in \mathbb{Z}^k$ and $w \in D(0;\delta)^{n-k}$.

Observe that, if we fix $\zeta \in D(1;\delta)$, $(z, \zeta\Phi(z) + w) \in \Omega$ $\forall z \in \overline{\mathbb{D}}^k$ and $\forall w \in D(0;\delta)^{n-k}$, owing to our choice of δ. Hence, we can define

$$H_\zeta^w(z) := f(z, \zeta\Phi(z) + w) \quad \forall z \in \overline{\mathbb{D}}^k,$$

where $\zeta \in D(1;\delta)$ and $w \in D(0;\delta)^{n-k}$. $H_\zeta^w(z) \in \mathcal{C}(\overline{\mathbb{D}}^k) \bigcap \mathcal{O}(\mathbb{D}^k)$. As $\mathbb{D}^k$ is a Reinhardt domain, by Theorem 1.3 , there exists a Laurent series expansion for H_ζ^w as follows:

$$H_\zeta^w(z) \ = \ \sum_{\alpha \in \mathbb{Z}^k} b_\alpha(w, \zeta) z^\alpha,$$

where $b_\alpha(w, \zeta)$ is uniquely determined by the following expression:

$$b_\alpha(w, \zeta) \ = \ \frac{1}{(2\pi i)^k} \int_{|z_1|=1} \cdots \int_{|z_k|=1} \frac{H_\zeta^w(z_1, z_2, ..., z_k)}{z_1^{\alpha_1+1} \cdots z_k{}^{\alpha_k+1}} dz_k...dz_1. \tag{3.2}$$

As $\mathbb{D}^k$ is a Reinhardt domain containing $0, b_\alpha(w, \zeta)$ vanishes for any $\alpha \in \mathcal{A} := \{\alpha \in \mathbb{Z}^k :$ $\alpha_j < 0$ for some $j = 1, ..., k\}$. But

$$
\begin{aligned}
&\frac{1}{(2\pi i)^k}\int_{|z_1|=1}\cdots\int_{|z_k|=1}\frac{H_\zeta^w(z_1,...,z_k)}{z_1^{\alpha_1+1}\cdots z_k{}^{\alpha_k+1}}dz_k...dz_1\\
=&\frac{1}{(2\pi i)^k}\int_{|z_1|=1}\cdots\int_{|z_k|=1}\frac{f(z_1,...,z_k,\zeta\Phi(z_1,...,z_k)+w)}{z_1^{\alpha_1+1}\cdots z_k{}^{\alpha_k+1}}dz_k...dz_1\\
=&\ G_\alpha^w(\zeta)\ \ \forall\zeta\in D(1;\delta)\text{ and }\alpha\in\mathbb{Z}^k.
\end{aligned}
\tag{3.3}
$$

From 3.2, 3.3 and our choice of w and ζ, we have $G_\alpha^w|_{D(1;\delta)}\equiv 0\quad\forall\alpha\in\mathcal{A}$ and $w\in D(0;\delta)^{n-k}$. We have shown that G_α^w is holomorphic in $D(0;1+\delta)\supseteq D(1;\delta)$. Hence, by the Identity Theorem, $G_\alpha^w\equiv 0$ $\forall\alpha\in\mathcal{A}$ and $w\in D(0;\delta)^{n-k}$. In particular, $G_\alpha^w(0)=0$ $\forall\alpha\in\mathcal{A}$ and $w\in D(0;\delta)^{n-k}$. Refer to (3.1) and observe that $G_\alpha^w(0)=a_\alpha(w)$ $\forall\alpha\in\mathbb{Z}^k$ and $w\in D(0;\delta)^{n-k}$. Hence, due to the holomorphicity of each a_α, we can apply the Identity Theorem to conclude that $a_\alpha\equiv 0$ $\forall\alpha\in\mathcal{A}$. Therefore, in fact,

$$f(z,w)\ =\ \sum_{\alpha\in\mathbb{N}^k}a_\alpha(w)z^\alpha$$

in $Ann(0;1-\varepsilon,1+\varepsilon)^k\times D(0;1+\varepsilon)^{n-k}$. Now, define $\tilde{f}:\mathbb{D}^n\to\mathbb{C}$ as

$$\tilde{f}(z,w)\ :=\sum_{\alpha\in\mathbb{N}^k}a_\alpha(w)z^\alpha$$

and observe that for each fixed w, the series on the right-hand side converges absolutely on $\mathbb{D}^{n-k}$. Hence, $\tilde{f}\in\mathcal{O}(\mathbb{D}^n)$ and $\tilde{f}\Big|_{Ann(0;1-\varepsilon,1+\varepsilon)^k\times D(0;1+\varepsilon)^{n-k}}\equiv f\Big|_{Ann(0;1-\varepsilon,1+\varepsilon)^k\times D(0;1+\varepsilon)^{n-k}}$. Since Ω is a connected neighbourhood of $Ann(0;1-\varepsilon,1+\varepsilon)^k\times D(0;1+\varepsilon)^{n-k}$, we conclude that $\tilde{f}$ is the required extension of f. □

The following result about the geometry of the envelope of holomorphy of the Ω described in the previous theorem is nearly immediate.

Corollary 3.2. *Let Ω be the domain described in Theorem 3.1. If $(\widetilde{\Omega},p)$ denotes the envelope of holomorphy of Ω, then $p(\widetilde{\Omega})$ contains $\mathbb{D}^n$.*

Proof. As $\Omega\cap\mathbb{D}^n$ is connected, $\{\Omega\cup\mathbb{D}^n, i:\Omega\hookrightarrow\Omega\cup\mathbb{D}^n\}$ is an $\mathcal{O}(\Omega)$-extension of (Ω,p_0), where $p_0:\Omega\cup\mathbb{D}^n\to\mathbb{C}^n$ is the inclusion map. Referring to Definition 2.3, we see that $\exists$ a holomorphic map $u:\Omega\cup\mathbb{D}^n\to X$ such that $p_0=p\circ u$. Hence $p(u(\mathbb{D}^n))=\mathbb{D}^n$. □

4. The Chirka-Rosay Extension Theorem

This chapter is devoted to studying a very surprising theorem by Chirka. For this purpose we need a definition. A *Hartogs figure in* $\mathbb{C}^n$ is a domain of the form that appears in Theorem 1.1.

Theorem 4.1 (Chirka, [5])**.** *Let* $\Phi \in \mathcal{C}(\overline{\mathbb{D}})$ *be such that* $\Phi(\partial\mathbb{D}) \subseteq \overline{\mathbb{D}}$ *and and satisfies* $\sup_{\overline{\mathbb{D}}} |\Phi| \leq 1$*. Let* Ω *be a connected neighbourhood of* $S := graph(\Phi) \cup (\partial\mathbb{D}) \times \overline{\mathbb{D}}$ *such that* $\Omega \cap \mathbb{D}^2$ *is connected. If* $(\widetilde{\Omega}, p)$ *denotes the envelope of holomorphy of* Ω*, then* $p(\widetilde{\Omega})$ *contains a Hartogs figure.*

We urge the reader to compare the above theorem with the folk theorem stated in Chapter 1 (i.e. Theorem 1.7). Its hypothesis would resemble that of Theorem 1.7 but for one stark difference: Φ above is merely continuous, *and is permitted to be extremely non-smooth.* This is what makes Chirka's theorem a very unexpected result. Referring to Chapter 3, we see that the proof of the folk theorem indeed proceeds by first showing that any $f \in \mathcal{O}(\Omega)$ extends holomorphically to a Hartogs figure. The conclusion of the above theorem is somewhat weaker than this. We will remark upon this difference at the end of this chapter.

The following proposition, which claims the existence of a solution to a particular non-linear $\bar{\partial}$-bar equation, is an essential ingredient in the proof of Theorem 4.1. The bulk of this chapter is devoted to its proof. This proposition was undertaken by Rosay and Chirka with the aim of simplifying Chirka's original proof in [5]. In that proof, Chirka worked with very different Banach spaces from those used below, which had resulted in much more complicated estimates.

Proposition 4.2. *Let* $\mathcal{F}$ *be the space of continuously differentiable functions defined on* $\mathbb{C}^2$*, with compact support. Then, for every* $\psi \in \mathcal{F}$*, there exists a unique* f *defined on* $\mathbb{C}$*, tending to* 0 *at infinity, which is a solution to*

$$\frac{\partial f}{\partial \bar{z}} = \psi(z, f(z)). \tag{4.1}$$

This solution depends continuously on $\psi \in \mathcal{F}$ *if the support of* ψ *is restricted to be in a given compact set, and if we use the* sup *norm for* f *and the* $\mathcal{C}^1$ *norm for* ψ*.*

The above proposition, in conjunction with Theorem 4.6, yields Theorem 4.1. In the proof that follows, a few ambiguities in Chirka and Rosay's proof have been resolved. For this, we need the following quantitative version of the Implicit Function Theorem (the gaps in Chirka and Rosay's proof seem to arise from the lack of a clear statement of the Implicit Function Theorem used therein).

Theorem 4.3. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be Banach spaces and $\Theta : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{E}_1$ be such that $\Theta \in \mathcal{C}^1(\mathbb{E}_1 \times \mathbb{E}_2)$. Suppose $a = (a_1, a_2) \in \mathbb{E}_1 \times \mathbb{E}_2$ is such that*

- $\Theta(a_1, a_2) = 0$,
- $\partial_1\Theta(a_1, a_2) : \mathbb{E}_1 \to \mathbb{E}_1$ *is invertible; and*
- *both* $\partial_1\Theta(a_1, a_2)$ *and* $[\partial_1\Theta(a_1, a_2)]^{-1}$ *are bounded.*

Let $\Lambda := \left\|[\partial_1\Theta(a_1, a_2)]^{-1}\right\|_{\mathbb{E}_1}$ *and* (δ_1, δ_2) *be so small that*

$$\left.\begin{array}{l}\|\partial_1\Theta(x) - \partial_1\Theta(a)\|_{\mathbb{E}_1} \\ \|\partial_2\Theta(x) - \partial_2\Theta(a)\|_{\mathbb{E}_2\to\mathbb{E}_1}\end{array}\right\} < \frac{1}{2\Lambda}$$

$\forall x \in B_{\mathbb{E}_1}(a_1; \delta_1) \times B_{\mathbb{E}_2}(a_2; \delta_2)$. *Then, there exists* $\theta : B_{\mathbb{E}_2}(a_2; \delta_2) \to \mathbb{E}_1$, $\theta \in \mathcal{C}^1(B_{\mathbb{E}_2}(a_2; \delta_2))$ *such that* $\theta(a_2) = a_1$ *and* $\Theta(\theta(x_2), x_2) = 0 \ \forall x_2 \in B_{\mathbb{E}_2}(a_2; \delta_2)$.

It is now time to fix, once and for all, the Banach space in which we will seek a solution to equation (4.1). For this, let

$$\mathcal{E} := \left\{ f \in \mathcal{C}_0(\mathbb{C}) : \frac{\partial f}{\partial \bar{z}} \text{ exists and belongs to } \mathcal{C}_0(\mathbb{C}) \text{ in the sense of distributions} \right\}.$$

The space $\mathcal{E}$ is equipped with the norm $\|f\|_{\mathcal{E}} = \sup(|f| + |\frac{\partial f}{\partial \bar{z}}|)$. It can be easily verified that $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space. Working with this space simplifies the problem as, for $f \in \mathcal{E}$, equation (4.1) is equivalent to

$$f = \frac{1}{\pi z} * \psi(z, f(z)). \tag{4.2}$$

To see this, let f_1 be a continuous solution of equation (4.2). As ψ is compactly supported in $\mathbb{C}^2$, and, consequently, $\widetilde{\psi} : z \mapsto \psi(z, f(z))$ is compactly supported in $\mathbb{C}$, we can set $K := \sup\{|y| : y \in supp(\widetilde{\psi})\}$. Now, for $|z| > K + n$,

$$\begin{aligned} |f_1(z)| &= \left| \int_{\mathbb{C}} \frac{1}{\pi(z-w)} \psi(w, f(w)) dA(w) \right| \\ &\leq \sup|\psi| \int_{supp(\widetilde{\psi})} \left| \frac{1}{\pi(z-w)} \right| dA(w) \qquad (4.3) \\ &\leq \frac{\sup|\psi|}{\pi n} A, \quad A \text{ is a constant independent of } \psi \text{ and } z. \end{aligned}$$

Hence, $|f_1(z)| \to 0$ as $|z| \to \infty$. Lastly, integrating the right-hand side of equation (4.2) against $\partial\phi/\partial\bar{z}$, where $\phi \in \mathcal{C}^\infty_C(\mathbb{C})$, yields $-\int_{\mathbb{C}} \psi(z, f(z))\phi(z)dA(z)$. This implies that any continuous solution of equation (4.2) belongs to $\mathcal{E}$ and is, indeed, a solution to equation (4.1) in the sense of distributions. Conversely, if $f_2 \in \mathcal{E}$ is a solution of (4.1), then it can differ from f_1 by a holomorphic function, say h. But, as both f_1 and f_2 vanish at infinity, $h \equiv 0$. Therefore, we can now shift our entire focus to equation (4.2), looking at which, it is possible to guess how the Implicit Function Theorem might have a role to play. We also need to choose a Banach space for ψ. Let

$$\mathcal{F}_0 := \{\psi \in \mathcal{C}^1_C(\mathbb{C}^2) : supp(\psi) \subset \mathbb{D}^2\}.$$

There is no loss of generality in restricting the support of ψ to the unit polydisc. We will require ψ in equation (4.1) to belong to $(\mathcal{F}_0, \|\cdot\|_1)$, where $\|\cdot\|_1$ is the $\mathcal{C}^1$ norm.

We are nearly ready to prove Proposition 4.2. The following result is an extremely crucial component.

Lemma 4.4. *Let $h \in \mathcal{E}$ be such that*

(i) $\dfrac{\partial h}{\partial \bar{z}}$ *has compact support ; and*

(ii) for some constant $C > 0$, $\left|\dfrac{\partial h}{\partial \bar{z}}\right| \leq C|h|$.

Then, $h \equiv 0$.

Proof. We use the method of integrating factors. The aim is to obtain a factor μ, bounded at infinity, such that μh is holomorphic in $\mathbb{C}$, i.e., $\dfrac{\partial(\mu h)}{\partial \bar{z}} = 0$. For this purpose, define

$$\lambda = -\frac{\partial h/\partial \bar{z}}{h}$$

at the points z where $h(z) \neq 0$, and (say) $\lambda(z) = 0$ if $h(z) = 0$. λ is bounded and has compact support due to *(ii)* and *(i)* respectively. Consequently, we can define $u := \frac{1}{\pi z} * \lambda$. We observe that

a) u is continuous,

b) $|u(z)| \to 0$ as $|z| \to 0$; and

c) $\dfrac{\partial u}{\partial \bar{z}} = \lambda$ in the sense of distributions.

a) is a consequence of the Dominated Convergence Theorem, while b) and c) can be proved by repeating arguments used while establishing the equivalence of equations (4.1) and (4.2). Set $\mu := e^u$. Then, $\partial(\mu h)/\partial\bar{z} = 0$ off the zero set of h. But μh is continuous everywhere and tends to 0 at infinity. Hence, by the Maximum Modulus Theorem — applied to μh on each connected component of $\mathbb{C} \setminus h^{-1}\{0\}$ — $h \equiv 0$. ∎

We will also employ the following estimate. It can be derived using, as in the proof of Lemma 4.4, the technique of integrating factors followed by the Maximum Principle. We just have to make a careful record of the magnitude of the relevant partial derivatives encountered. Since this estimation is essentially routine, we leave the details for the reader to fill in.

Lemma 4.5. *For every $M > 0$, there exists $C_M > 0$ such that, for every $\xi \in \mathcal{E}$, and every continuous function χ with support in the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ satisfying $|\chi(z)| \leq M|\xi(z)|$, we have*

$$\left\| \xi - \frac{1}{\pi z} * \chi \right\|_{\mathcal{E}} \geq C_M \|\xi\|_{\mathcal{E}}.$$

Proof of Proposition 4.2. **Uniqueness.** Assume $f,\ g \in \mathcal{E}$ such that $\partial f/\partial \bar{z} = \psi(z, f(z))$ and $\partial g/\partial \bar{z} = \psi(z, g(z))$. Set $h = f - g$. Clearly, $h \in \mathcal{E}$ and $\partial h/\partial \bar{z} = \psi(z, f(z)) - \psi(z, g(z))$ is compactly supported. Also, as $\psi \in \mathcal{C}^1_C(\mathbb{C}^2)$, $\exists R > 0$ such that $|\psi(z, f(z)) - \psi(z, g(z))| \leq R|f(z) - g(z)|$. Hence, by Lemma 4.4, $h \equiv 0$, i.e., $f \equiv g$.

Existence. It suffices to prove the following claim:

($*$) *For every $M > 0$, there exists $\varepsilon_M > 0$ such that if $\psi \in \mathcal{F}_0$ with $\|\psi\|_1 \leq M$, and if equation (4.2) is solvable (with f in $\mathcal{E}$), then for every $\psi' \in \mathcal{F}_0$ satisfying $\|\psi - \psi'\|_1 \leq \varepsilon_M$, the equation $f' = \frac{1}{\pi z} * \psi'(z, f'(z))$ is solvable, and f' depends continuously on ψ'.*

For, if we are seeking a solution to equation (4.2) for $\psi \in \mathcal{F}_0$, we let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \frac{\varepsilon_M}{\|\psi\|_1}$, where ε_M corresponds to $M = \|\psi\|_1$ as in ($*$). Letting $\widetilde{\psi}_j = \frac{j}{n}\psi,\ j \leq n$, we connect, in finitely many steps, ψ to 0. Equation (4.2) is trivially solvable for 0.

The problem has now been reduced to a claim which facilitates the application of the Implicit Function Theorem mentioned above (Theorem 4.3). We define $\Theta : \mathcal{E} \times \mathcal{F}_0 \to \mathcal{E}$ as

$$\Theta : (f, \psi) \mapsto f - \frac{1}{\pi z} * \psi(z, f(z)).$$

Let $M > 0$ and $\psi^0 \in \mathcal{F}_0$ be such that $\|\psi^0\|_1 \leq M$ and $\exists f^0 \in \mathcal{E}$ such that $\partial f^0/\partial \bar{z} = \psi^0(z, f^0(z))$, i.e., $\Theta(f^0, \psi^0) = 0$. If Θ and $a = (f^0, \psi^0)$ satisfy the conditions of Theorem 4.3, then we obtain a δ_2 such that equation (4.2) is solvable for all $\psi \in B_{\mathcal{F}_0}(\psi^0; \delta_2)$. This proves ($*$) but for one obstruction. We emphasise here that, in general, the $\delta_2 > 0$ in Theorem 4.3 will depend on the point a and the value of the corresponding Λ. Our aim is, therefore, to show that:

1) Θ and $a = (f^0, \psi^0)$ satisfy all the hypotheses of Theorem 4.3; and

2) δ_2 can be made indepedent of the point $a = (f, \psi)$, as long as $\|\psi\| \leq M$.

Remark. Observe that δ_1 is free to depend on the point a. This important observation will be exploited in the pursuit of our second aim.

Step 1. We now itemise and prove the various components of our first aim. We keep in mind that the $\overline{\partial}$-derivative of the function $f - \frac{1}{\pi z} * \psi(z, f(z))$ is $\partial f/\partial \bar{z} - \psi(z, f(z))$.

a) $\boldsymbol{\Theta \in \mathcal{C}^1(\mathcal{E} \times \mathcal{F}_0)}$: Evaluating Θ at $(f + \xi, \psi)$, where ξ is an infinitesimal increment, leads to the differential

$$\partial_1\Theta(f,\psi)(\xi) = \xi - \frac{1}{\pi z} * [\psi_w(z, f(z))\xi(z) + \psi_{\bar{w}}(z, f(z))\overline{\xi(z)}]. \tag{4.4}$$

Similarly, evaluating Θ at $(f, \psi + \eta)$, η an infinitesimal increment, we obtain

$$\partial_2\Theta(f,\psi)(\eta) = \frac{1}{\pi z} * \eta(z, f(z)). \tag{4.5}$$

The continuity of $\Theta, \partial_1\Theta$ and $\partial_2\Theta$ in both the variables is manifest.

b) $\boldsymbol{\partial_1\Theta(f^0, \psi^0)}$ **is bounded**: For the sake of convenience, let, for $\xi \in \mathcal{E}$,

$$\alpha^0_\xi(z) := \psi^0_w(z, f^0(z))\xi(z) + \psi^0_{\bar{w}}(z, f^0(z))\overline{\xi(z)}, \; z \in \mathbb{C}.$$

We see that $|\alpha^0_\xi(z)| \leq \|\xi\|_{\mathcal{E}}\|\psi^0\|_1$. Thus,

$$\begin{aligned}
\|\partial_1\Theta(f^0,\psi^0)(\xi)\|_{\mathcal{E}} =& \sup_{w\in\mathbb{C}} \left\{ \left|\xi(w) - \left(\frac{1}{\pi z} * \alpha^0_\xi(z)\right)(w)\right| + \left|\frac{\partial\xi}{\partial\bar{z}}(w) - \alpha^0_\xi(w)\right| \right\} \\
\leq& \|\xi\|_{\mathcal{E}} \left[1 + \|\psi^0\|_1 \int_{\mathbb{D}} \left|\frac{1}{\pi z}\right| dA(z) + \|\xi\|_{\mathcal{E}}(1 + \|\psi^0\|_1)\right]. \\
\leq& \|\xi\|_{\mathcal{E}} \left[2 + M\left(1 + \int_{\mathbb{D}} \left|\frac{1}{\pi z}\right| dA(z)\right)\right].
\end{aligned}$$

Hence $\partial_1\Theta(f^0, \psi^0)$ is bounded and its operator norm is bounded above by a constant, say C_1, depending only on M.

c) $\boldsymbol{\partial_1\Theta(f^0, \psi^0)}$ **is invertible**: Let $\Xi : \mathcal{E} \to \mathcal{E}$ be the operator

$$\xi \mapsto \frac{1}{\pi z} * [\psi^0_w(z, f^0(z))\xi(z) + \psi^0_{\bar{w}}(z, f^0(z))\overline{\xi(z)}] = \frac{1}{\pi z} * \alpha^0_\xi(z).$$

We claim that Ξ is compact. For this, let $V \subset \mathcal{E}$ be open and bounded and $U := \Xi(V)$. Let $L > 0$ be such that $\|\zeta\|_{\mathcal{E}} \leq L \; \forall \zeta \in U$ (L exists as Ξ is bounded). Define $\widetilde{U} := \{\phi \in \mathcal{C}^1_0(\mathbb{C}) \cap \mathcal{E} : \|\phi\|_{\mathcal{E}} \leq L+1\}$. Being a bounded set in $(\mathcal{C}^1_0(\mathbb{C}) \cap \mathcal{E}, \|\cdot\|_{\mathcal{E}})$, $\widetilde{U}$ is equicontinuous as a consequence of a standard estimate that involves the generalised Cauchy integral formula and exploits the uniform bound on the $\partial/\partial\bar{z}$-derivatives. Now, for any $\zeta \in U$, as $\partial\zeta/\partial\bar{z} \in \mathcal{C}_0(\mathbb{C})$ in the sense of distributions, there exists a sequence $\{\widetilde{\phi}_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}_C(\mathbb{C})$ such that $\sup|\widetilde{\phi}_n - \partial\zeta/\partial\bar{z}| \to 0$ as $n \to \infty$. Now, let $\phi_n := \frac{1}{\pi z} * \widetilde{\phi}_n,\ n \in \mathbb{N}$. Then, $\phi_n \in \mathcal{C}_0^1(\mathbb{C}) \cap \mathcal{E}$ and ϕ_n converges *uniformly* to ζ as $n \to \infty$. Without loss of generality, as $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ζ in the $\|\cdot\|_{\mathcal{E}}$ norm, $\|\phi_n\|_{\mathcal{E}} \leq L+1,\ n \in \mathbb{N}$. Thus, every $\zeta \in U$ is the uniform limit of some sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $\widetilde{U} \subset \mathcal{C}_0^1(\mathbb{C}) \cap \mathcal{E}$. As equicontinuity is preserved under the action of taking uniform limits, U is also equicontinuous. Further, since $\alpha_\xi^0(z)$ is compactly supported on $\mathbb{C}$, we repeat arguments used earlier (refer to (4.3)) to obtain estimates as follows:

$$|z| > 1+n \Rightarrow |\Xi(\xi)(z)| \leq \frac{A\|\xi\|_{\mathcal{E}}\|\psi\|_1}{\pi n} \leq \frac{AL\|\psi\|_1}{\pi n} \ \ \forall \xi \in V.$$

Thus, U is bounded, equicontinous and given $\varepsilon > 0$, there exists a compact $K \subset \mathbb{C}$ such that $|\zeta(z)| < \varepsilon\ \forall z \notin K$ and $\forall \zeta \in U$. This tells us that U is totally bounded and, hence, pre-compact in $(\mathcal{C}_0(\mathbb{C}) \cap \mathcal{E}, \|\cdot\|_\infty)$. It is, as a consequence, easily verified that U is, in fact, pre-compact in $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$.

Coming back to $\partial_1\Theta(f^0,\psi^0)$, we have just shown that it is the perturbation of the identity by a compact operator. So $\partial_1\Theta(f^0,\psi^0)$ is a Fredholm operator of index 0. This means that dim[ker$\partial_1\Theta(f^0,\psi^0)$] =dim[coker$\partial_1\Theta(f^0,\psi^0)$] and the range of $\partial_1\Theta(f^0,\psi^0)$ is closed. Thus, to achieve (c), it is sufficient to show that $\Theta(f^0,\psi^0)$ is injective. Now, if $\partial_1\Theta(f^0,\psi^0)(\xi) = 0$, i.e.,

$$\xi = \frac{1}{\pi z} * \alpha_\xi^0(z)$$

then, then, it is easy to conclude that ξ satisfies the hypotheses of Lemma 4.1. Therefore, $\xi \equiv 0$, whence $\partial_1\Theta(f^0,\psi^0)$ is invertible.

d) $[\boldsymbol{\partial_1\Theta(f^0,\psi^0)}]^{-1}$ **is bounded**: As $\partial_1\Theta(f^0,\psi^0)$ is surjective, any $\zeta \in \mathcal{E}$ can be written as $\xi - \frac{1}{\pi z} * \alpha_\xi^0(z)$ for some $\xi \in \mathcal{E}$, i.e., $[\partial_1\Theta(f^0,\psi^0)]^{-1}(\zeta) = \xi$. Also, $|\alpha_\xi^0(z)| \leq \|\psi^0\|_{\mathcal{F}_0}|\xi(z)| \leq M|\xi(z)|$. Hence, by Lemma 4.5,

$$\left\|[\partial_1\Theta(f^0,\psi^0)]^{-1}(\zeta)\right\|_{\mathcal{E}} \leq C_2\|\zeta\|_{\mathcal{E}},$$

where $C_2 = \frac{1}{C_M}$ depends only on M.

Step 2. To achieve our second goal, we first observe that $\partial_1\Theta$ is uniformly continuous in the second variable, while $\partial_2\Theta$ is altogether independent of it. Therefore, $\exists\varepsilon_M > 0$ such that

$$\|\psi' - \psi\|_1 < \varepsilon_M \Rightarrow \|\partial_1\Theta(f,\psi') - \partial_1\Theta(f,\psi)\|_{\mathcal{E}} < \frac{1}{4C_2},$$

for all $f \in \mathcal{E}$ and $\psi \in \mathcal{F}_0$ such that $\|\psi\|_1 \leq M$. Now, let $\nu^0 > 0$ be such that

$$\|f - f^0\|_{\mathcal{E}} < \nu^0 \Rightarrow \left.\begin{array}{l} \|\partial_1\Theta(f,\psi^0) - \partial_1\Theta(f^0,\psi^0)\|_{\mathcal{E}} \\ \|\partial_2\Theta(f,\psi^0) - \partial_2\Theta(f^0,\psi^0)\|_{\mathcal{F}_0\to\mathcal{E}} \end{array}\right\} < \frac{1}{4C_2}$$

Thus, $\forall(f,\psi) \in B_{\mathcal{E}}(f^0;\nu^0) \times B_{\mathcal{F}_0}(\psi^0;\varepsilon_M)$,

$$\begin{aligned} &\|\partial_1\Theta(f,\psi) - \partial_1\Theta(f^0,\psi^0)\|_{\mathcal{E}} \leq \|\partial_1\Theta(f,\psi) - \partial_1\Theta(f,\psi^0)\|_{\mathcal{E}} + \|\partial_1\Theta(f,\psi^0) - \partial_1\Theta(f^0,\psi^0)\|_{\mathcal{E}} \\ &\leq \frac{1}{2C_2} \leq \frac{1}{2\|[\partial_1\Theta(f^0,\psi^0)]^{-1}\|_{\mathcal{E}}}, \end{aligned}$$

and

$$\|\partial_2\Theta(f,\psi) - \partial_2\Theta(f^0,\psi^0)\|_{\mathcal{F}_0\to\mathcal{E}} = \|\partial_2\Theta(f,\psi^0) - \partial_2\Theta(f^0,\psi^0)\|_{\mathcal{F}_0\to\mathcal{E}} \leq \frac{1}{2\|[\partial_1\Theta(f^0,\psi^0)]^{-1}\|_{\mathcal{E}}}.$$

Hence by Theorem 4.3, there exists $\theta : B_{\mathcal{F}_0}(\psi^0;\varepsilon_M) \to \mathcal{E}$, $\theta \in \mathcal{C}^1(B_{\mathcal{F}_0}(\psi^0;\varepsilon))$ such that $\theta(\psi^o) = f^0$ and $\Theta(\theta(\psi),\psi) = 0$ $\forall\psi \in B_{\mathcal{F}_0}(\psi^0;\varepsilon_M)$, i.e., equation (4.2) is solvable for ψ, a solution being $\theta(\psi)$, $\forall\psi \in B_{\mathcal{F}_0}(\psi^0;\varepsilon_M)$. But, since ε_M depends only on M, the above proof can be repeated for any $\psi \in \mathcal{F}_0$ as long as $\|\psi\|_1 \leq M$ and $f = \frac{1}{\pi z} * \psi(z, f(z))$ is solvable with f in $\mathcal{E}$. □

The above proof does not work in higher dimensions. It breaks down when one tries to imitate Lemma 4.4. If we were solving this problem in $\mathbb{C}^3$, for instance, we would require a solution, in some subspace of $\mathcal{C}_0(\mathbb{C};\mathbb{C}^2)$, to $\overline{\partial} f = \psi$, where $\psi \in \mathcal{C}^1_C(\mathbb{C}^3;\mathbb{C}^2)$. But it is not true that

$$\left|\frac{\partial h_1}{\partial \bar{z}}\right| \leq C(|h_1| + |h_2|) \text{ and } \left|\frac{\partial h_2}{\partial \bar{z}}\right| \leq C(|h_1| + |h_2|),$$

and vanishing at infinity, imply $h_1 = h_2 = 0$. For this, take $h_1 = h_2 = 1/z$ for $|z| > 1$, and extend them in the unit disc such that h_1 and h_2 never vanish simultaneously. The required inequalities clearly hold for all $C > 0$ outside the unit disc. Within the unit disc, we can obtain a bound on $\dfrac{|\partial h_i/\partial \bar{z}|}{|h_1| + |h_2|}$, $i = 1, 2$.

The next key result is a paraphrasing of a result known as the Chirka-Stout Kontinuitätssatz that is usable in the situation of our interest.

Theorem 4.6 (Chirka-Stout, [7]). *Let X be a domain of holomorphy in $\mathbb{C}^2$. Let Ω be a subdomain of X and let $D \Subset \Omega$ be a relatively compact open subset. Suppose $\Psi : \mathbb{D}\times[0,1] \to X$ is a continuous function with the following properties:*

- *For each $t \in [0,1]$, the set $\psi_t := \Psi(\mathbb{D} \times \{t\}) \setminus \overline{D}$ is a complex-analytic subvariety of $X \setminus \overline{D}$.*
- *There exists a $t^0 \geq 0$ such that $\psi_{t^0} \neq \emptyset$ and such that $\psi_t \subset \Omega$ $\forall t \leq t^0$.*

Let $(\widetilde{\Omega}, p)$ *denote the envelope of holomorphy of* Ω*. Then* $\psi_1 \subset p(\widetilde{\Omega})$.

We want to emphasise that the original Chirka-Stout Kontinuitätssatz is much more general than the above statement, and examines the relation between continuous families of holomorphic p-chains and the envelopes of holomorphy of domains of arbitrary dimensions. Its proof involves some *sophisticated facts about complex-analytic subvarieties that I am not currently familiar with* (e.g. Bishop's Theorem on the limits of analytic sets, Wirtinger's Inequality, etc.). *Hence, Theorem 4.6 will be used without proof.*

Proof of Theorem 4.1. Let Φ and Ω be as given. Let $\varepsilon > 0$ be so small that $Ann(0; 1-2\varepsilon, 1+2\varepsilon) \times D(0; 1+\varepsilon) \subset \Omega$. Construct $\widetilde{\Phi} : \mathbb{C} \to \mathbb{C}$ such that it has the following properties:

* $\widetilde{\Phi} \in \mathcal{C}^1(\mathbb{C})$,
* $|\widetilde{\Phi}(z) - \Phi(z)|$ is so small, for $|z| < 1-\varepsilon$, that $graph(\widetilde{\Phi}|_{D(0;1-\varepsilon)}) \subset \Omega$; and
* $\widetilde{\Phi}(z) = 0, \ \forall z$ such that $|z| \geq 1$.

Define $\widetilde{S} := graph(\widetilde{\Phi}) \cup \partial\mathbb{D} \times \overline{\mathbb{D}}$. By construction, $\widetilde{S} \subset \Omega$. Let U and D be two open sets in Ω satisfying:

$$\widetilde{S} \subset U \subset\subset D \subset\subset \Omega \cap \big(D(0; 1+\varepsilon) \times \mathbb{C}\big).$$

Let $\chi \in \mathcal{C}^\infty_C(\mathbb{C}^2)$ with

* $\chi^{-}\{1\} = U$; and
* $supp(\chi) \subset D$.

Finally, define the continuous family of functions $\{F_t : t \in [0,1]\} \subset \mathcal{C}^1_C(\mathbb{C}^2)$ by the equation

$$F_t(z, w) := (1-t)\chi(z, w)\frac{\partial\widetilde{\Phi}}{\partial\bar{z}}(z) \qquad \forall (z, w) \in \mathbb{C}^2.$$

Consider the family of PDE's:

$$\frac{\partial f_t}{\partial \bar{z}}(z) = F_t(z, f_t(z)) \qquad (f_t \in \mathcal{E}). \tag{4.6}$$

Here, $\mathcal{E}$ is as in the proof of Proposition 4.2. Note that

$$supp(F_s) = supp(F_t) \qquad \forall s, \ t \in [0,1].$$

Thus by Proposition 4.2, (4.6) admits a unique solution in $\mathcal{E}$ for each $t \in [0,1]$ and these solutions vary continuously (w.r.t. the *sup* norm). By uniqueness (in $\mathcal{E}$),

$$f_0 = \widetilde{\Phi}, \text{ and} \tag{4.7}$$

$$f_1 = 0. \tag{4.8}$$

Now, let $X := D(0;1+2\varepsilon) \times \mathbb{C}$ and $\mathfrak{D} := \Omega \cap X$. Note, by construction that $D \subset\subset \mathfrak{D}$. Define $\Psi : \mathbb{D} \times [0,1] \to X$ by

$$\Psi(\zeta, t) := \big((1+2\varepsilon)\zeta, f_t((1+2\varepsilon)\zeta)\big).$$

It is clear that $\Psi(\mathbb{D} \times \{t\}) = graph\left(f_t\big|_{D(0;1+2\varepsilon)}\right)$, i.e., these are submanifolds of X; and that Ψ is continuous. Lastly, pick a (z_0, w_0) in $X \setminus \overline{D}$ such that (z_0, w_0) lies on $\Psi(\mathbb{D} \times \{t\})$ for some t. $\exists \delta > 0$ such that $\mathfrak{B}\big((z_0, w_0); \delta\big) \subset X \setminus \overline{D}$. Let

$$\omega := \pi_z\big(\mathfrak{B}\big((z_0, w_0); \delta\big) \cap \Psi(\mathbb{D} \times \{t\})\big),$$

where $\pi_z :=$ the projection onto the z-axis. Note that

$$\xi \in \omega \Rightarrow \frac{\partial f_t}{\partial \bar{z}}(\xi) = (1-t)\chi(\xi, f_t(\xi))\frac{\partial \widetilde{\Psi}}{\partial \bar{z}}(\xi) = 0.$$

Since $\mathfrak{B}\big((z_0, w_0); \delta\big) \cap \Psi(\mathbb{D} \times \{t\}) = graph(f_t) \cap \mathfrak{B}\big((z_0, w_0); \delta\big)$, we have thus shown that:

$$\psi_t \neq 0 \Rightarrow \psi_t \text{ is a complex-analytic submanifold of } X \setminus \overline{D} \tag{4.9}$$

where $\psi_t := \Psi(\mathbb{D} \times \{t\}) \setminus \overline{D}$. In other words, all the requirements of the Chirka-Stout Kontinuitätsatz are met (with $\mathfrak{D}$ playing the role of Ω in Theorem 4.6). So, if $(\widetilde{\mathfrak{D}}, \pi)$ denotes the envelope of holomorphy of $\mathfrak{D}$, then owing to (4.7)–(4.9),

$$[\psi_1 \cup (\mathfrak{D} \cap \{w = 0\})] \cup Ann(0; 1-2\varepsilon, 1+2\varepsilon) \times D(0;1) \subset \pi(\widetilde{\mathfrak{D}}).$$

Since $\pi(\widetilde{\mathfrak{D}})$ is an open set, $\exists r > 0$ small enough, such that the Hartogs figure

$$\mathbb{H} := \big(D(0;1) \times D(0;r)\big) \cup \big(Ann(0; 1-2\varepsilon, 1+2\varepsilon) \times D(0;1)\big) \subset \pi(\widetilde{\mathfrak{D}}). \tag{4.10}$$

Now, note that $f \in \mathcal{O}(\Omega) \Rightarrow f\big|_{\mathfrak{D}} \in \mathcal{O}(\mathfrak{D})$, whence every $f \in \mathcal{O}(\Omega)$ extends to $\widetilde{\mathfrak{D}}$, i.e., $(\widetilde{\mathfrak{D}}, \pi)$ is and $\mathcal{O}(\Omega)$-extension of Ω. Thus, there exists an analytic $u : \widetilde{\mathfrak{D}} \to \widetilde{\Omega}$ such that $\pi = p \circ u$. Now,

$$p(\widetilde{\Omega}) \supseteq p \circ u(\widetilde{\mathfrak{D}}) = \pi(\widetilde{\mathfrak{D}})).$$

Hence, in view of (4.10), $p(\widetilde{\Omega})$ contains the Hartogs figure $\mathbb{H}$. $\square$

In a widely circulated preprint of Chirka's theorem, the conclusion of Theorem 4.1 was stated thus, " If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^2$." (Compare with Theorem 1.7). However, this stronger conclusion was found not to be supported by Chirka's arguments. Intuitively: when two complex-analytic sets ψ_s and ψ_t, $s \neq t$ (in the notation of the above theorem) intersect at a point — which is *not* ruled out by Chirka's methods — multi-valuedness of the attempted extension of f can result.

In [5], Chirka also asks if Theorem 4.1 is valid in the multidimensional case — i.e., when $\Phi := \left(\Phi_1, ..., \Phi_n\right)$ is a continuous $\mathbb{D}^n-$valued map with $n > 1$. Rosay [14] showed that the theorem fails, in general, for higher dimensions. Thereafter, several attempts were made to address Chirka's question for vector-valued maps with component functions belonging to a proper sub-class of $\mathcal{C}(\mathbb{D}; \mathbb{C})$. In Chapters 6 and 7, we describe the underlying techniques of some successful attempts in this direction.

5. Merker and Porten's Proof of the Hartogs Extension Theorem

In this chapter, we move away from Hartogs-Chirka type of configurations to examine another well-known result by Hartogs which states that all holomorphic functions in a connected neighbourhood $\mathcal{V}(\partial\Omega)$ of $\partial\Omega \Subset \mathbb{C}^n$, $n \geq 2$, extend holomorphically and uniquely to the domain Ω. Recall that we have already seen a proof of this in Chapter 1 (Theorem 1.5).

Often, analytic-continuation results rely on some variation of the *method of analytic discs* — a technique for achieving analytic continuation of a holomorphic function $f \in \mathcal{O}(\Omega_1)$ to a larger domain Ω_2, that involves extending the function along continuously varying analytic discs/varieties which eventually fill up Ω_2 but remain attached to Ω_1 along their borders. However, the standard proof of Theorem 1.5 that we saw in Chapter 1 uses no such ideas. The main challenge in rigorously proving this result using merely the tool of analytic discs lies in establishing the single-valuedness of the extension. Understanding this is our motivation to revisit the Hartogs phenomenon described above. In their paper [11], Merker and Porten successfully tame the issue of multisheetedness by using the method of analytic discs for local extensional steps and some Morse-theoretic tools for the global topological control of monodromy. Several details of their proof are technical and elaborate, and hence, cannot be presented here. But, we would like to present a broad outline of their ideas. We first state the extension theorem.

Theorem 5.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain having connected boundary. If $n \geq 2$, every function holomorphic in some connected open neighbourhood $\mathcal{V}(\partial\Omega)$ of $\partial\Omega$ extends holomorphically and uniquely inside Ω, i.e.,*

$$\forall f \in \mathcal{O}(\mathcal{V}(\partial\Omega)),\ \exists \text{ a unique } F \in \mathcal{O}(\Omega \cup \mathcal{V}(\partial\Omega)) \text{ such that } F\big|_{\mathcal{V}(\partial\Omega)} \equiv f.$$

An outline of Merker and Porten's proof

Some remarks on the notation used in this proof:

(i) For any given $E \subset \mathbb{C}^n$, $E_{>r}$ will denote the set $E \cap \{z : ||z|| > r\}$. In some places, the subscript "$> r$" is used for objects E which are not globally defined. In those cases, the relevance of "$> r$" will be contextually clear. For instance, for a given hypersurface M,

$M_{>r} := M \cap \{z : ||z|| > r\}$, while the components of $M_{>r}$ are labeled as $M^c_{>r}$, $1 \leq c \leq c_\lambda$. M^c, however, is not globally defined.

(ii) The domains bound by $M^c_{>r}$ and $\{z : ||z|| = r\}$ are denoted by $\widetilde{\Omega}^c_{>r}$. Here, the tilde notation indicates that $\widetilde{\Omega}^c_{>r}$ may not be contained in the the domain Ω_M bound by M.

(iii) For any subset $E \subset \mathbb{C}^n$ and $\delta > 0$, $\mathcal{V}_\delta(E) := \cup_{p \in E} \mathbb{B}^n(p, \delta)$ denotes the tubular neighbourhood of E with cross-sectional radius δ.

- *Step 1*: We perturb $\partial\Omega$ to a $\mathcal{C}^\infty-$smooth connected oriented hypersurface $M \Subset \mathcal{V}(\partial\Omega)$ for which the restriction to M of the Euclidean-norm function $z \mapsto ||z||$ is a Morse function with only finitely many non-degenerate critical points $\widehat{p}_\lambda \in M$, $1 \leq \lambda \leq \kappa$, with $||\widehat{p}_1|| < \cdots < ||\widehat{p}_\kappa||$. If $\mathcal{V}(\partial\Omega)$ is a thin tubular neighbourhood $\mathcal{V}_\delta(M)$ contained in $\mathcal{V}(\partial\Omega)$, with cross-section $0 < \delta \ll 1$, then $\mathcal{V}(\partial\Omega)$ and Ω in Theorem 5.1 can be replaced by $\mathcal{V}(\partial\Omega)$ and $\Omega_M :=$ the domain enclosed by M, repsectively.
- *Step 2*: Let $\widehat{r}_\lambda := ||\widehat{p}_\lambda||$. For any arbitrary fixed radius r with $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$, and some fixed λ with $1 \leq \lambda \leq \kappa - 1$, consider all the connected components $M^c_{>r}$, $1 \leq c \leq c_\lambda$, of the cut-out hypersurface $M \cap \{||z|| > r\}$. Their number c_λ is the same for all $r \in (\widehat{r}_\lambda, \widehat{r}_{\lambda+1})$. We show that each connected hypersurface $M^c_{>r} \subset \{z : ||z|| > r\}$ bounds a certain domain $\widetilde{\Omega}^c_{>r} \subset \{z : ||z|| > r\}$ where

$$\widetilde{\Omega}^c_{>r} := \text{the domain bound by } M^c_{>r} \text{ and } \{z : ||z|| = r\} \text{ that is relatively compact in } \mathbb{C}^n.$$

One subtlety: to retain connectedness, we must consider a slightly modified neighbourhood $\mathcal{V}_\delta(M_{>r})_{>r}$ of $M_{>r}$ instead of considering the neighbourhood $\mathcal{V}_\delta(M) \cap \{z : ||z|| > r\}$ of $M_{>r}$ (we will elaborate on this later in this chapter).
- *Step 3*: Now consider a modification of the Hartogs figure, called the *Levi-Hartogs figure*, defined as follows:

$$\mathcal{LH}_{\varepsilon_1,\varepsilon_2} := \left\{ \max_{1 \leq i \leq n-1} |z_i| < \varepsilon_1,\ |x_n| < \varepsilon_1,\ -\varepsilon_2 < y_n < 0 \right\} \bigcup \left\{ \varepsilon_1 - (\varepsilon_1)^2 < \max_{1 \leq i \leq n-1} |z_i| < \varepsilon_1,\ |x_n| < \varepsilon_1,\ |y_n| < \varepsilon_2 \right\}.$$

By computing the Cauchy integral on appropriate analytic discs whose boundaries remain in $\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$, we conclude that holomorphic functions in this (bed-like) figure extend holomorphically to the full parallelopiped

$$\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}} := \left\{ \max_{1 \leq i \leq n-1} |z_i| < \varepsilon_1,\ |x_n| < \varepsilon_1,\ |y_n| < \varepsilon_2 \right\}.$$

The Levi-Hartogs figure is used to produce holomorphic extension from cut-out domains

$\{z : ||z|| > r\} \cap \Omega_M$ such that the radius r can be reduced by a uniform amount after which the same procedure is repeated. In other words, we can deduce analytic continuation to Ω_M by induction.

- *Step 4*: To handle the phenomenon of multivaluedness effectively, we need to deal with the components $M^c_{>r}$, $1 \leq c \leq c_\lambda$, separately. The following proposition talks about analytic continuation in $\widetilde{\Omega}^c_{>r}$.

  Proposition 5.2. *Fix a radius r satisfying $\widehat{r}_\lambda < r < \widehat{r}_{\lambda+1}$, for some λ with $1 \leq \lambda \leq \kappa - 1$. Then, for each $c = 1, ..., c_\lambda$, and for each function holomorphic in $\mathcal{V}_\delta(M_{>r})_{>r}$, its restriction to a neighbourhood $\mathcal{V}_\delta(M^c_{>r})_{>r}$ of $M^c_{>r}$ extends holomorphically and uniquely to $\widetilde{\Omega}^c_{>r}$ by means of a finite number of Levi-Hartogs figures.*

  Main ideas in the Proof: For filling the top of the domain Ω_M— i.e., for $\lambda = \kappa - 1$ — we observe that the single component $\widetilde{\Omega}_{>r} =: \Omega_{>r}$, $r \in (\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$, is diffeomorphic to a cut-out piece of the ball. Placing Levi-Hartogs figures successively (as we shall see later), we can descend from r to $r - \eta$ (as long as $r - \eta > \widehat{r}_{\kappa-1}$) for some uniform η with $0 < \eta \ll 1$ that depends on the dimension $n \geq 2$, on δ and on the diameter of Ω.

  For descending below $\widehat{r}_{\kappa-1}$, we need an inductive procedure that helps us

  A: fill the domains through intervals of the form (r', r'') such that $\widehat{r}_\lambda < r' < r'' < \widehat{r}_{\lambda+1}$, $\lambda = 1, ..., \kappa - 2$; and

  B: jump across singular radii.

For A, we can show that certain advantageous topological properties hold for every one of the cut-out domains associated to (r', r''). For instance, the region bounded by the hypersurface M and any two spheres of radii r' and r'', with $r', r'' \in (\widehat{r}_\lambda, \widehat{r}_{\lambda+1})$, is diffeomorphic to a finite union of tube-like domains. Placing Levi-Hartogs figures successively yields holomorphic extension along these tube-like domains. Another important property is that two different domains $\widetilde{\Omega}^{c_l}_{>r}$ and $\widetilde{\Omega}^{c_k}_{>r}$ are either disjoint or one is contained in the other. Consequently, multivaluedness will occur only if $\widetilde{\Omega}^{c_l}_{>r} \subset \widetilde{\Omega}^{c_k}_{>r}$ (or vice-versa) and two uniquely defined holomorphic extensions $f^{c_l}_r$ to $\widetilde{\Omega}^{c_l}_{>r}$ and $f^{c_k}_r$ to $\widetilde{\Omega}^{c_k}_{>r}$ differ on $\widetilde{\Omega}^{c_l}_{>r}$.

Accomplishing B proves to be more complicated as, unlike in the case A, the collection of domains $\widetilde{\Omega}^c_{>r}$ may undergo significant topological changes. Owing to the nature of Morse functions, three different topological processes may occur:

i. *creation* of a new component $\widetilde{\Omega}^{c'}_{>r-\eta}$;

ii. *merger* of two components $\widetilde{\Omega}^{c_1}_{>r}$ and $\widetilde{\Omega}^{c_2}_{>r}$ into $\widetilde{\Omega}^{c'}_{>r-\eta}$; and

iii. *disappearance* of some component $\widetilde{\Omega}^c_{>r}$ with the property $\widetilde{\Omega}^c_{>r} \not\subset \Omega_M$.

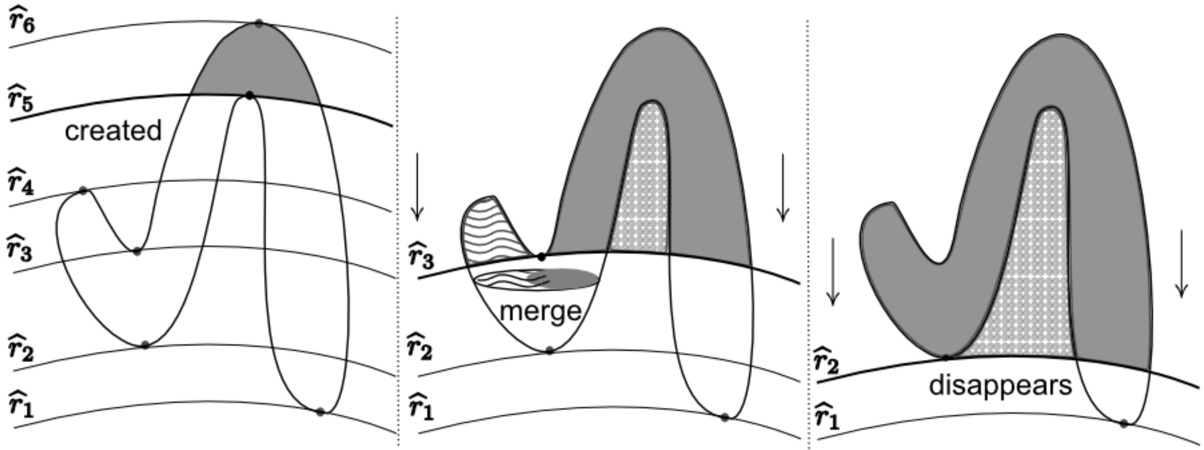


FIGURE 1 Creation, merger and disappearance of components.[1]

The type of topological change, and hence, the procedure required to analytically continue the holomorphic function in question depends on the Morse index k_λ of the singularity $\widehat{p}_\lambda$. The proof is further divided into three steps corresponding to the cases $k_\lambda = 0$ and $2n-1$, $2 \leq k_\lambda \leq 2n-2$ and $k_\lambda = 1$. It is in the last case that the question of suppressing the analytic continuation determined by a disappearing component arises. Merker and Porten devise an easy, yet efficient, tool to keep track of the components which need to be subtracted to avoid mulitvaluedness. □

- *Step 5*: Finally, we apply Proposition 5.2 to the single component $\widetilde{\Omega}_{>\widehat{r}_1+\varepsilon}$ ($\varepsilon \ll \delta$) to obtain the conclusion of the extension theorem.

Preparation of the boundary and unique extension

In the above outline, the first step — the preparation of a 'good' Morse boundary — allows us to control the global topology of the cut-out domains $\Omega_M \cap \{z : ||z|| > r\}$. We now describe this construction. Let $\delta_1 > 0$ be so small that the tubular neighbourhood $\mathcal{V}_{\delta_1}(\partial\Omega) := \cup_{p\in\partial\Omega}\mathbb{B}^n(p,\delta_1)$ lies entirely in the initial neighbourhood $\mathcal{V}(\partial\Omega)$. Then, choosing a point $p_0 \in \mathbb{C}^n$ such that $dist(p_0, \overline{\Omega}) = 3$, center the coordinates $(z_1, ..., z_n)$ at p_0. Consider the function $r(z) : z \mapsto ||z||$. By standard results in Morse theory ([10, Chapter 6, Theorem 1.2]), we can find a $\mathcal{C}^\infty-$smooth, connected and oriented hypersurface $M \subset \mathcal{V}_{\delta_1/2}(\partial\Omega)$ such that $r_M(z) := r(z)\big|_M$ is a Morse function with only finitely many non-degenerate critical points $\widehat{p}_\lambda \in M,\ 1 \leq \lambda \leq \kappa$, and M bounds a unique domain Ω_M with $\Omega \subset \Omega_M \cup \mathcal{V}(\partial\Omega)$. Moreover, using transversality arguments, the point p_0 can be chosen in such a way that the critical points of r_M lie on different level sets of r, i.e., $2 \leq r_M(\widehat{p}_1) < \cdots < r_M(\widehat{p}_\kappa)$. Such an M is called a *good boundary.*

[1]Illustration taken from Merker and Porten [11], Section 1.

We now need to verify that the Hartogs theorem can be reduced to proving a version involving a good boundary M replacing $\partial\Omega$ (and Ω_M replacing Ω).

Lemma 5.3. *Suppose that for some δ with $0 < \delta \leq \delta_1/2$ so small that $\mathcal{V}_\delta(M)$ is a thin tubular neighbourhood of the good boundary M, the Hartogs theorem holds for the pair $(\Omega_M, \mathcal{V}_\delta(M))$. Then, the Hartogs extension property holds for the given pair $(\Omega, \mathcal{V}(\partial\Omega))$.*

Proof. Let $f \in \mathcal{O}(\mathcal{V}(\partial\Omega))$. Then, the restriction of f to $\mathcal{V}_\delta(M)$ admits an extension $F_\delta \in \mathcal{O}(\Omega_M \cup \mathcal{V}_\delta(M))$ by hypothesis. As $\Omega \subset \Omega_M \cup \mathcal{V}(\partial\Omega)$, it is enough to show that $\Omega_M \cap \mathcal{V}(\partial\Omega)$ is connected. This is because f and F_δ already coincide in $\mathcal{V}_\delta(M) \cap \Omega_M \subset \mathcal{V}(\partial\Omega)$.

Let $p,\ q \in \Omega_M \cap \mathcal{V}(\partial\Omega)$. Then, there exists a $\mathcal{C}^\infty-$smooth curve $\gamma : [0,1] \to \mathcal{V}(\partial\Omega)$ connecting p to q. If $\mathsf{Image}(\gamma) \subset \Omega_M$, we are done. If not, then $\mathsf{Image}(\gamma)$ must cross M. If $\mathsf{Image}(\gamma)$ meets M, let p' be the first point on $\mathsf{Image}(\gamma) \cap M$ and let q' be the last one. Now, modify γ by joining p' to q' by means of a curve μ entirely contained in the connected hypersurface M. Now pushing μ slightly inside Ω_M one gets an appropriate curve running from p to q inside $\Omega_M \cap \mathcal{V}(\partial\Omega)$. Thus, $\Omega_M \cap \mathcal{V}(\partial\Omega)$ is connected. Now to complete the proof, define the required extension as

$$F := \begin{cases} F_\delta, & \text{in } \Omega_M \cup \mathcal{V}_\delta(M), \\ f, & \text{in } \mathcal{V}(\partial\Omega). \end{cases}$$

□

Here, we would like to remark that several modern approaches to analytic-continuation problems rely upon making such "admissible" changes to the geometry of the given configuration. Such a move is quite essential when Ω has — unlike the theorems in Chapter 1 — very few symmetries. We will see instances of this in the following chapters (Chapters 6 and 7).

In view of the above lemma, we must — in order to prove Theorem 5.1 — show that the pair $(\Omega_M, \mathcal{V}_\delta(M))$ has the Hartogs extension property.

Global Levi-Hartogs filling from the farthest point

In order to demonstrate Merker and Porten's method of analytic discs for local analytic continuation, we will summarize the main ideas that go into filling up the domain (with Levi-Hartogs figures) from the farthest critical point $\widehat{p}_\kappa$ to the next critical point $\widehat{p}_{\kappa-1}$. This will also shed some light on the procedure employed to fill up the domain through intervals of the form (r', r'') such that $\widehat{r}_\lambda < r' < r'' < \widehat{r}_{\lambda+1}$, $\lambda = 1, ..., \kappa - 2$. The procedure for extending any $f \in \mathcal{O}(\mathcal{V}_\delta(M))$ to the portion of Ω_M lying between two level sets lying on either sides of a critical level set $\{z : ||z|| = \widehat{r}_\lambda\}$, $\lambda = 1, .., \kappa - 1$, is very technical, and breaks up into several cases. It will not be possible to do justice to this procedure in this report. However, the procedure for extending $f \in \mathcal{O}\big(\mathcal{V}_\delta(M)\big)$ in the portion of Ω_M lying above the level set $\{z : ||z|| = \widehat{r}_{\kappa-1}\}$ already reveals several key techniques. These are the techniques we shall discuss.

(i) *Preparing the Levi-Hartogs figure*: It is first important to understand how the Levi-Hartogs figure introduced in Step 3 in the above outline of Merker and Porten's proof can be used in our situation. We will use the following notation:

- For $r \in \mathbb{R}$ with $r > 1$ and $\delta \in \mathbb{R}$ with $0 < \delta \ll 1$, let $\mathcal{S}_r^{r+\delta} := \{r < ||z|| < r + \delta\}$.
- For a $R \subset S_r^{2n-1} := \{z \in \mathbb{C}^n : ||z|| = r\}$ open in the relative topology of S_r^{2n-1}, we define the (radial) *rind of thickness* $\eta > 0$ *around* R as $\mathsf{Rind}(R, \eta) := \{(1 + s)z : z \in R, |s| < |\eta|/r\}$.
- For any subset $E \subset S_r^{2n-1}$ and $\delta \in \mathbb{R}$ with $0 < \delta \ll 1$, define $\mathsf{Shell}_r^{r+\delta}(E) := \cup_{p\in E}\mathbb{B}^n(p, \delta) \cap \{z : ||z|| > r\}$.

We now observe that, given $\delta > 0$ and $p \in S_r^{2n-1}$, we can find $\varepsilon_1,\ \varepsilon_2$ and some composition of a translation and a unitary map, say Φ_p, that sends the origin to p, the real-tangent plane $T_0\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$ to the real-tangent plane $T_pS_r^{2n-1}$ and the whole of $\mathcal{LH}_{\varepsilon_1,\varepsilon_2}$ inside the shell $\mathcal{S}_r^{r+\delta}$. Additionally, Φ_p can be chosen in such a manner that $\Phi_p(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}})$ contains a rind of thickness $c\frac{\delta^2}{r}$ around some region $R_p \subset S_r^{2n-1}$ whose $(2n-1)$-dimensional area only depends on δ. From this fact, we can deduce the following crucial proposition:

Proposition 5.4. *Let $R \subset S_r^{2n-1}$ (with $r > 1$ and $n \geq 2$) be a relatively open set having $\mathcal{C}^\infty-$smooth boundary $N = \partial R$ and let $\delta \in \mathbb{R}$ with $0 < \delta \ll 1$. Then, holomorphic functions in $\mathsf{Shell}_r^{r+\delta}(R \cup N)$ extend holomorphically to a rind of thickness $c\frac{\delta^2}{r}$ around R by means of a finite number of Levi-Hartogs figures.*

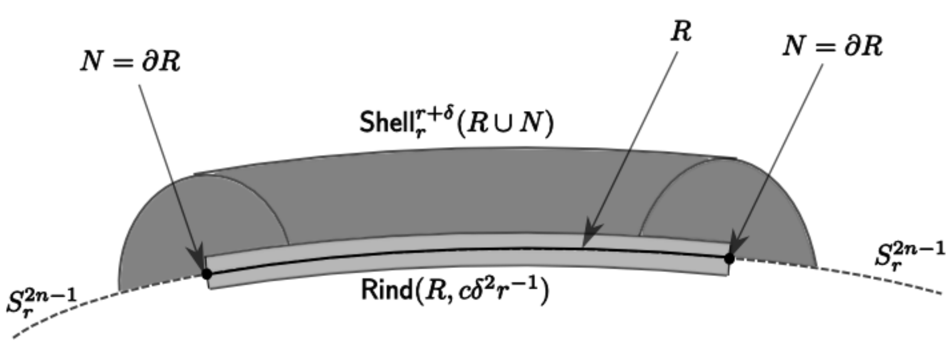


FIGURE 2 Extensions from a pseudoconcave piece of shell.[2]

[2]Illustration taken from Merker and Porten [11], Section 3.2.

Here, one chooses finitely many points $p_1, ..., p_m \in R \cup N$ such that the associated local regions R_{p_k} contained in the filled Levi-Hartogs figures $\Phi_{p_k}(\widehat{\mathcal{LH}_{\varepsilon_1,\varepsilon_2}})$ cover $R \cup N$. Then, one extends the function on each Levi-Hartogs figure separately and patches up the functions thus obtained into a single holomorphic function by establishing connectedness of relevant intersections of open sets.

(ii) *The geometry of the cut-out hypersurface* $M_{>r}$, $r \in (\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$: By assumption, the real-Hessian matrix of r_M is nondegenerate at $\widehat{p}_\kappa$ and the tangency of $\partial\mathbb{B}^n(0, \widehat{r}_\kappa)$ to $M = \partial\Omega_M$ at $\widehat{p}_\kappa$ forces strong convexity of M at $\widehat{p}_\kappa$. Basic Morse theory shows that $M_{>r}$ is a deformed spherical cap diffeomorphic to $\mathbb{R}^{2n-1}$ for every $r \in (\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$. Also, $\Omega_{>r} = \Omega_M \cap \{z : ||z|| > r\}$ is then a piece of deformed ball diffeomorphic to $\mathbb{R}^{2n}$. The boundary in $\mathbb{C}^n$ of $\Omega_{>r}$

$$\partial\Omega_{>r} = M_{>r} \cup R_r \cup N_r$$

consists of $M_{>r}$ together with the open subregion $R_r := \Omega_M \cap \{||z|| = r\}$ of S_r^{2n-1} which is diffeomorphic to $\mathbb{R}^{2n-1}$ and has boundary $N_r := M \cap \{||z|| = r\}$ diffeomorphic to the unit $(2n-2)$-sphere.

(iii) *Choosing the neighbourhood* $\mathcal{V}_\delta(M_{>r})_{>r}$: One might naively wish to consider the open set $\mathcal{V}_\delta(M)_{>r}$. However, when $r > \widehat{r}_{\kappa-1}$ is very close to $\widehat{r}_{\kappa-1}$, a connected component $\mathcal{W}_{>r}$ of $\mathcal{V}_\delta(M)_{>r}$ might appear above $\{r = \widehat{r}_{\kappa-1}\}$ (as seen in the diagram below). After filling $\Omega_{>r}$ progressively, by means of Levi-Hartogs figures, because $\Omega_{>r} \cap \mathcal{V}_\delta(M)_{>r}$ is *not* connected — the extension of f thus created will not, in general, agree with $f\big|_{\mathcal{W}_{>r}}$.

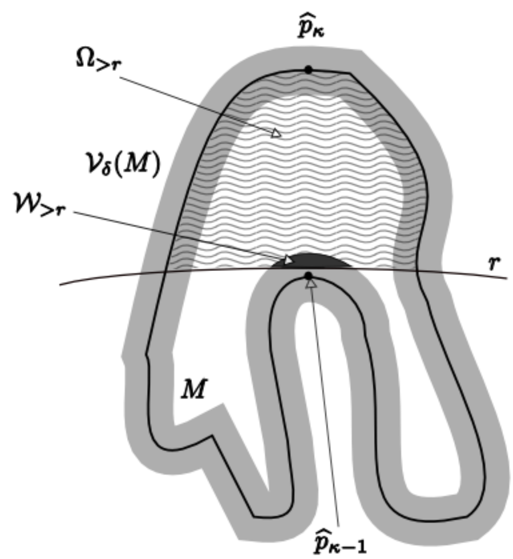


FIGURE 3 Occurence of problematic components.[3]

[3]Illustration taken from Merker and Porten [11], Section 4.1.

To erase such problematic components $\mathcal{W}_{>r}$, we will consider the open set

$$\mathcal{V}_\delta(M_{>r})_{>r} = \mathcal{V}_\delta(M_{>r}) \cap \{z : ||z|| > r\}.$$

This open set is always diffeomorphic to $M_{>r} \times (-\delta, \delta)$.

(iv) *Choosing* η: For the rinds appearing in Propisition 5.4, we use the smallest appearing thickness

$$\eta := \frac{c\delta^2}{\widehat{r}_\kappa},$$

and, if required, further shrink η to ensure that $\eta \ll \delta$. It is important to note that, given the hypotheses of Proposition 5.4, the conclusion of the proposition can be obtained for any rind around R of thickness less than $c\delta^2/r$.

(v) *Filling up* $\Omega_{>r}$ *using shells*: To prove Proposition 5.2 for $\lambda = \kappa - 1$, fix a radius $r \in (\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$. By placing a single Levi-Hartogs figure at $\widehat{p}_\kappa$, we get unique holomorphic extension to $\Omega_{>\widehat{r}_\kappa - \eta}$. As $\eta \ll \delta$, $\widehat{r}_\kappa - \eta > \widehat{r}_{\kappa-1}$. If the radius $\widehat{r}_\kappa - \eta < r$, then shrinking the rind, we get a unique extension to $\Omega_{>r}$. If not, then performing induction on an auxiliary integer $k \geq 1$, we suppose that, by descending from $\widehat{r}_\kappa$ to a lower radius $r' := \widehat{r}_\kappa - k\eta \geq r$, holomorphic functions in $\mathcal{V}_\delta(M_{>r})_{>r}$ extend uniquely to $\Omega_{>r'}$. Now, we wish to descend further to $\Omega_{>r'-\eta}$. In view of Proposition 5.4, we are required to show that, for every radius r' with $\widehat{r}_{\kappa-1} < r < r' < \widehat{r}_\kappa$,

- $\mathsf{Shell}_{r'}^{r'+\delta}(R_{r'} \cup N_{r'}) \subset \Omega_{>r'} \cup \mathcal{V}_\delta(M_{>r})_{>r}$;
- $\mathsf{Rind}(R_{r'}, \eta) \bigcup (\Omega_{>r'} \cup \mathcal{V}_\delta(M_{>r})_{>r}) \supset \Omega_{>r'-\eta} \cup \mathcal{V}_\delta(M_{>r})_{>r}$; and
- $\mathsf{Rind}(R_{r'}, \eta) \bigcap (\Omega_{>r'} \cup \mathcal{V}_\delta(M_{>r})_{>r})$ is connected.

The first claim allows us to extend the function $f_{r'}$ (the unique extension to $\Omega_{r'}$ whose existence is guaranteed by our induction hypothesis) to $\mathsf{Rind}(R_{r'}, \eta)$. The second claim helps us construct a potential candidate for the extension function to $\Omega_{>r'-\eta}$, and the third claim ensures that this candidate is indeed an extension. The proofs of these steps are extremely geometric in nature and Merker and Porten often resort to pictures for the proofs. The third claim involves decomposing the rind into three parts, each of which is dealt with separately. Although we will not go into the details of the proof, we would like to point out that the ideas heavily depend on our understanding of the geometry of the hypersurface M and the cut-out domains $\Omega_{>r}$, $r \in (\widehat{r}_{\kappa-1}, \widehat{r}_\kappa)$.

Finally, by induction, we can conclude that $\exists k_0 \in \mathbb{N}$ such that $\widehat{r}_\kappa - (k_0 - 1)\eta > r$, $\widehat{r}_{\kappa-1} < \widehat{r}_\kappa - k_0\eta \leq r$ and that we get a unique holomorphic extension to $\Omega_{>\widehat{r}_\kappa - k_0\eta}$. If $r' - k_0\eta < r$, then we shrink the thickness of the final extensional rind to obtain a unique and holomorphic extension to $\Omega_{>r}$, as required.

6. A Generalization of Chirka's Extension Theorem

We now return to Hartogs-Chirka type of configurations to explore the validity of Chirka's extension theorem (Chapter 4, Theorem 4.1) in higher dimensions. Recall that in Chapter 4, we mentioned that Rosay's counterexample in [14] gave a negative answer to Chirka's question $\big($[5, Question 1], also see Chapter 4$\big)$. In this chapter, we present Bharali's ([2]) first generalization of Chirka's extension theorem to higher dimensions:

Theorem 6.1 (Bharali, [2])**.** *Let Γ be the graph of the map $\phi = (\phi_1, ..., \phi_n) : \overline{\mathbb{D}} \to \mathbb{C}^n$, where for each ϕ_k, $k = 1, ..., n$,*

$$\phi_k \in \left[\left\{ z \mapsto \psi(z, \overline{z}) : \psi \in \mathcal{O}(\mathbb{D}^2) \text{ and } \sup_{(z,\zeta)\in\mathbb{D}^2} |\psi(z,\zeta)| < 1 \right\} \cap \mathcal{C}(\overline{\mathbb{D}}; \mathbb{D}) \right]. \tag{6.1}$$

If Ω is a connected neighbourhood of $S := \Gamma \cup \big(\partial\mathbb{D} \times \overline{\mathbb{D}}^n\big)$ such that $\Omega \cap \mathbb{D}^{n+1}$ is connected and if $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^{n+1}$.

We first consider the following lemma — a special case of Theorem 6.1 — which is not only crucial to the proof of Theorem 6.1, but also illustrates a powerful technique often used in analytic-continuation problems. The kind of construction seen in the proof of this lemma will be repeated more than once in the final chapter of this report.

Lemma 6.2. *Let Γ be the graph of the function $\phi(z) := \overline{z}$ over $\overline{\mathbb{D}}$. Let Ω be a neighbourhood of $\Gamma \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$ such that $\Omega \cap \mathbb{D}^2$ is connected. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to a neighbourhood of $\mathbb{D}^2$.*

Proof. Consider the smooth family of analytic discs $\{\mathcal{A}_t\}_{t\in[-1,1]}$:

$$\mathcal{A}_t(z) := (z, 2t - z), \quad z \in \overline{\mathbb{D}}.$$

Notice that $\mathcal{A}_t(\overline{\mathbb{D}}) \cap \Gamma = \{(t + iy, t - iy) : y \in [-\sqrt{1-t^2}, \sqrt{1-t^2}]\}$.

Consider the case when $t \geq 0$. If $t \leq x \leq 1$ and $x^2 + y^2 \leq 1$, then $|2t - (x + iy)| \leq 1$. Thus, the sets

$$\mathcal{S}_t^+ := \left\{ \mathcal{A}_t(x + iy) : t < x < 1,\ x^2 + y^2 < 1 \right\},\ t \in [0, 1),$$

are analytic discs with boundaries in Ω, and $\mathcal{S}_t^+ \subset \Omega$ for t close to 1. By the Kontinuitätssatz, f extends holomorphically to a neighbourhood of $\{(z = x+iy, -z) \in \mathbb{C}^2 : 0 \leq x \leq 1,\ |z| \leq 1\}(= \overline{\mathcal{S}_0^+})$.

Similarly, by considering the sets

$$\mathcal{S}_t^- := \left\{\mathcal{A}_t(x+iy) : -1 < x < t,\ x^2+y^2 < 1\right\},\ t \in (-1, 0],$$

we can show that f extends holomorphically to a neighbourhood of the graph of the holomorphic function $\psi(z) = -z$. Call this function $\widetilde{f}$. Now we can evoke the classical Hartogs theorem (i.e., Theorem 3.1) to conclude that $\widetilde{f}$ extends holomorphically to $\mathbb{D}^2$. $\square$

We now proceed to prove the main theorem. Note that the class of graphs described in Theorem 6.1 is such that the 'Γ-component' of S may , in general, be entirely devoid of symmetry. We will first make — just as in the proof of Theorem 5.1 — admissible changes to the configuration S so that it suffices to prove Theorem 6.1 for ϕ_k belonging to some easier-to-work-with class of graphs. Thereafter, the conjugate variable $\overline{z}$ is treated as an independent variable ξ.

Proof of Theorem 6.1. Let $\phi : z \mapsto (\psi_1(z,\overline{z}), ..., \psi_n(z,\overline{z}))$ be a map belonging to the class described in (6.1) above. We then choose an $\varepsilon > 0$ such that $Ann(0; 1-2\varepsilon, 1+2\varepsilon) \times \overline{\mathbb{D}}^n \subset \Omega$. It is easy to see that it suffices to work with the Hartogs configuration $\mathsf{graph}\left(\phi|_{\overline{D(0;1-\varepsilon)}}\right) \cup \left(\partial D(0; 1-\varepsilon) \times \overline{\mathbb{D}}^n\right)$. But then, we have the very useful property:

$$(z, \xi) \mapsto (\psi_1(z,\xi), ..., \psi_n(z,\xi)) \text{ is continuous on } \overline{D(0; 1-\varepsilon)}^2.$$

Therefore, it actually suffices to prove Theorem 6.1 under the assumption that for each ϕ_k, $k = 1, ..., n$,

$$\phi_k \in \left\{ z \mapsto \psi(z,\overline{z}) : \psi \in \mathcal{O}(\mathbb{D}^2) \cap \mathcal{C}(\overline{\mathbb{D}}^2) \text{ and } \sup_{(z,\zeta)\in\overline{\mathbb{D}}^2} |\psi(z,\zeta)| < 1 \right\}. \tag{6.2}$$

Moreover, we can let ψ_k, $k = 1, ..., n$, be polynomials $P_k(z,\zeta) = \sum_{|\alpha|=0}^{N} C_\alpha^{(k)} z^{\alpha_1}\zeta^{\alpha_2}$, with $\sup_{(z,\zeta)\in\overline{\mathbb{D}}^2} |P(z,\zeta)| < 1$. This is beacuse ψ_k can now be approximated arbitrarily closely by such polynomials on the whole of $\overline{\mathbb{D}}^2$. Hence, we will now assume that ϕ_k, $k = 1, ..., n$ are polynomials $\phi_k(z) = P_k(z,\overline{z})$ with the property $\sup_{(z,\zeta)\in\overline{\mathbb{D}}^2} |P_k(z,\zeta)| < 1$.

Let $\delta > 0$ be so small that

i) $\sup_{(z,\zeta)\in\overline{D(0;1+\delta)}^2} |P_k(z,\zeta)| < 1$;

ii) $\{(z, w_1, ..., w_n) \in \mathbb{C}^{n+1} : |z| < 1, |w_k - P_k(z,\overline{z})| < \delta;\ k = 1, ..., n\} \subset \Omega$; and *iii)* $Ann(0;1-\delta+1+\delta)\times\overline{\mathbb{D}}^n\subset\Omega$.

There exists an $\varepsilon_0\in(0,\delta/2)$ so small that for $|w_k|<\varepsilon_0$, $k=1,...,n$,

- $|\{w_k+P_k(z,\xi)\}-P_k(z,\overline{z})|<\delta \quad \forall(z,\xi)\in\mathbb{D}\times\mathbb{C}$ such that $|\xi-\overline{z}|<\varepsilon_0$; and
- $|w_k+P_k(z,\xi)|<1 \quad \forall(z,\xi)\in Ann(0;1-\delta,1+\delta)\times D(0;1+\delta)$.

Therefore, for each $w\in D(0;\varepsilon_0)^n$

$$(z,\xi)\mapsto H(z,\xi,w):=f\big(z,w_1+P_1(z,\xi),...,w_n+P_n(z,\xi)\big) \tag{6.3}$$

is well-defined and holomorphic in $\{(z,\xi):|z|<1,|\xi-\overline{z}|<\varepsilon_0\}\cup(Ann(0;1-\delta,1+\delta)\times D(0;1+\delta))\subset\mathbb{C}^2$. Hence, by Lemma 6.2, we can define a function $\widetilde{H}$, such that for each $w\in D(0;\varepsilon_0)^n$,

$$(z,\xi)\mapsto\widetilde{H}(z,\xi,w)$$

is a holomorphic extension of the function given by (6.3) to a neighbourhood of $\mathbb{D}^2$.

Now, observe that for any $(z_0,\xi_0)\in\mathbb{D}^2$ and $\mu\in\big(|\xi_0|,1\big)$,

$$\begin{aligned}\widetilde{H}(z_0,\xi_0,w) &= \frac{1}{(2\pi i)^2}\int\limits_{|z|=1}\int\limits_{|\xi|=\mu}\frac{\widetilde{H}(z,\xi,w)}{(z-z_0)(\xi-\xi_0)}d\xi dz\\ &= \frac{1}{(2\pi i)^2}\int\limits_{|z|=1}\int\limits_{|\xi|=\mu}\frac{H(z,\xi,w)}{(z-z_0)(\xi-\xi_0)}d\xi dz\end{aligned}$$

for $w\in D(0;\varepsilon_0)^n$. Thus, the analyticity of the family $\big\{H(\cdot,\cdot,w)\big\}_{|w_k|<\varepsilon_0}$ forces $\Big\{\widetilde{H}(\cdot,\cdot,w)\Big\}_{|w_k|<\varepsilon_0}$ to be an analytic family. We now have that

$$(z,\xi,w)\mapsto\widetilde{H}(z,\xi,w_1-P_1(z,\xi),...,w_n-P_n(z,\xi))$$

is holomorphic in $\{(z,\xi,w):|z|<1,|\xi|<1,|w_k-P_k(z,\xi)|<\varepsilon_0\}$.

Define

$$\widetilde{f}(z,w):=\widetilde{H}(z,0,w_1-P_1(z,0),...,w_n-P_n(z,0)).$$

$\widetilde{f}$ is defined and holomorphic in $\{(z,w):|z|<1,|w_k-P_k(z,0)|<\varepsilon_0\}$. Since, for $z\in Ann(0;1-\delta,1+\delta)$, $\widetilde{H}(z,0,w_1-P_1(z,0),...,w_n-P_n(z,0))=H(z,0,w_1-P_1(z,0),...,w_n-$

$P_n(z,0))$, we have

$$\begin{aligned}\widetilde{f}(z,w_1,...,w_n) &= \widetilde{H}(z,0,w_1-P_1(z,0),...,w_n-P_n(z,0)) \\ &= H(z,0,w_1-P_1(z,0),...,w_n-P_n(z,0)) \\ &= f(z,w_1,...,w_n) \quad \text{[by (6.3)]}\end{aligned}$$

in $R_\delta := \{(z,w) : z \in Ann(0;1-\delta,1+\delta), |w_k - P_k(z,0)| < \varepsilon_0\}$. So, $\widetilde{f}\big|_{R_\delta} \equiv f\big|_{R_\delta}$. This allows us to define $\widetilde{f}$ as a holomorphic function on $A_\delta := Ann(0;1-\delta,1+\delta) \times \mathbb{D}^n$ by simply setting $\widetilde{f}\big|_{A_\delta} = f\big|_{A_\delta}$. We, therefore, conclude that $\widetilde{f}$ is holomorphic in a neighbourhood of $\sigma \cup (\partial D \times \mathbb{D}^n)$ — σ being the graph of the holomorphic map $z \mapsto (P_1(z,0),...,P_n(z,0))$ over $\overline{\mathbb{D}}$. By the classical theorem of Hartogs, $\widetilde{f}$ extends to $F \in \mathcal{O}(\mathbb{D}^{n+1})$ and $F\big|_{R_\delta} \equiv f\big|_{R_\delta}$. R_δ being an open subset of Ω, F is the required holomorphic extension. □

Although a valid generalization of Chirka's extension theorem, Theorem 6.1 deals with a subclass of $\mathcal{C}(\overline{\mathbb{D}};\mathbb{D}^n)$ that is quite restrictive. We observed that, by changing the scheme of choosing analytic discs in the proof of Lemma 6.2, we could prove this theorem for a less restrictive class. We talk about this in the next chapter. We will also see another subclass of $\mathcal{C}(\overline{\mathbb{D}};\mathbb{D}^n)$ for which such a theorem has been obtained by Barrett and Bharali ([1]).

7. Two Extension Theorems of Hartogs-Chirka Type Involving Continuous Multifunctions

Introduction and statement of results

This chapter is motivated by a version of Hartogs' lemma that says that if Ω is some neighbourhood of the union of $\partial\mathbb{D} \times \mathbb{D}$ and a complex analytic subvariety $\Sigma \subset \overline{\mathbb{D}} \times \mathbb{D}$ that is finitely-sheeted over $\mathbb{D}$ (such that $\Omega \cap \mathbb{D}^2$ is connected), and $f \in \mathcal{O}(\Omega)$, then f continues holomorphically to $\mathbb{D}^2$; and by the Hartogs-type extension theorem of Chirka (Chapter 4, Theorem 4.1). One is motivated to ask whether, given the following "Weierstrass pseudopolynomial"

$$\mathcal{P}_a(z,w) := w^k + \sum_{j=0}^{k-1} a_j(z) w^j, \ k \geq 2, \tag{7.1}$$

where $a_0, ..., a_{k-1} \in \mathcal{C}(\overline{\mathbb{D}})$, with $\mathcal{P}_a^{-1}\{0\} \subset \overline{\mathbb{D}} \times \mathbb{D}$, and a neighbourhood Ω of $\mathcal{P}_a^{-1}\{0\} \cup \big(\partial\mathbb{D} \times \overline{\mathbb{D}}\big)$, the conclusion of the aforementioned theorems can be inferred in this new setting.

One possible approach to this question is suggested by the Kontinuitätssatz-based strategies of Bharali [2] and Barrett-Bharali [1], provided one is willing to allow $(a_0, ..., a_{k-1})$ in (7.1) to belong to some strict subclass of $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$. To motivate the origins of the two main theorems below, let us recall the result from [2] that we saw in the previous chapter.

Result 7.1 (Bharali, [2])**.** *Let Γ be the graph of the map $(\phi_1, ..., \phi_k) : \overline{\mathbb{D}} \to \mathbb{C}^k$, each $\phi_j(z) := \psi_j(z, \overline{z})$, where, for $j = 1, ..., k$,*

$$\psi_j \in \left\{ \psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(z,\zeta) \in \overline{\mathbb{D}}^2} |\psi(z,\zeta)| < 1 \textit{ and } z \mapsto \psi(z, \overline{z}) \textit{ is continuous on } \overline{\mathbb{D}} \right\}. \tag{7.2}$$

If Ω is a connected neighbourhood of $S := \Gamma \cup (\partial\mathbb{D} \times \mathbb{D}^k)$ contained in $\{(z,w) \in \mathbb{C} \times \mathbb{C}^k : w \in \overline{\mathbb{D}}^k\}$ and if $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^{k+1}$.

In the theorems in [1] and [2], the authors construct a continuous family of discs $\big\{\Phi_t \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k) : t \in [0,1]\big\}$ such that $\Phi_0 = (\phi_1, ..., \phi_k)$ and each Φ_t is holomorphic on larger and larger sub-regions of $\mathbb{D}$ so that, eventually, $\Phi_1 \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. This suggests the following strategy:

- *Step 1.* Setting $(\phi_1, ..., \phi_k) := (a_0, ..., a_{k-1})$, we can try to construct a continuous family of discs $\{\Phi_t\}_{t\in[0,1]}$ with the properties mentioned above. We can then treat each $\Phi_t := (\Phi_{t,0}, ..., \Phi_{t,k-1})$ as a k-tuple of the ordered coefficients of a Weierstrass pseudopolynomial, to obtain a continuous family of "pseudovarieties" $\left\{\Sigma_t := \left\{(z, w) \in \mathbb{D} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} \Phi_{t,j}(z)w^j = 0\right\}\right\}_{t\in[0,1]}$ such that $\Sigma_0 := \{(z, w) \in \overline{\mathbb{D}} \times \mathbb{C} : \mathcal{P}_a(z, w) = 0\}$, each Σ_t is a finitely-sheeted complex analytic subvariety fibered over larger and larger sub-regions of $\mathbb{D}$, and Σ_1 is the graph of an analytic multifunction (i.e., a multigraph) over $\mathbb{D}$.

- *Step 2.* In the above construction, our hypotheses on $(a_0, ..., a_{k-1})$ must also ensure that each Σ_t over $\mathbb{D}$, like the initial "pseudovariety", lies within the bidisc, i.e., $\Sigma_t \subset \overline{\mathbb{D}} \times \mathbb{D}\ \forall t \in [0, 1]$, and that Σ_t is attached to $\partial\mathbb{D} \times \mathbb{D}$ along the border of $\Sigma_t\ \forall t \in [0, 1]$.

- *Step 3.* Finally, we invoke a suitable version of the Kontinuitätssatz to achieve analytic continuation along the family constructed above so as to reduce the problem to the finitely-sheeted-analytic-variety version of Hartogs' lemma mentioned in the beginning of this section.

It turns out that this second strategy is successful (with some refinement) if the coefficients $a_0, ..., a_{k-1}$ are drawn from the subclasses studied in [1] and [2]. The results presented below are contained in the article [8]. The first theorem is stated for $a_0, ..., a_{k-1}$ belonging to the subclass of $\mathcal{C}(\overline{\mathbb{D}})$ introduced by Barrett and Bharali in [1].

Theorem 7.2. *Let $a_0, ..., a_{k-1} \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C})$ be such that the set*

$$\Sigma_a := \left\{(z, w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z)w^j = 0\right\}$$

lies entirely in $\overline{\mathbb{D}} \times \mathbb{D}$. For $0 < r \leq 1$, let $A^j_\nu(r)$ represent the ν^{th} Fourier coefficient of $a_j(re^{i\cdot})$, $\nu \in \mathbb{Z}$. Assume that $A^j_\nu \equiv 0\ \forall \nu < 0$ and $j = 0, ..., k - 1$. Let Ω be a connected neighbourhood of $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$ such that $\Omega \cap \mathbb{D}^2$ is connected. Then, for every $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega\cap\mathbb{D}^2} \equiv f\big|_{\Omega\cap\mathbb{D}^2}.$$

Our next theorem has its origins in Result 7.1, but see Remarks 7.1 and 7.2 below.

Theorem 7.3. *Let $a_j := \psi_j(z, \overline{z})$, where*

$$\psi_j \in \left\{\psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(\zeta,s)\in\mathbb{D}\times[0,1]} |\psi(\zeta, s\overline{\zeta})| < 1 \text{ and } z \mapsto \psi(z, \overline{z}) \text{ is continuous on } \overline{\mathbb{D}}\right\} \quad (7.3)$$

Chapter 7: Extension Theorems of Hartogs-Chirka Type Chapter 7: Extension Theorems of Hartogs-Chirka Type

for $j = 0, ..., k-1$, *be such that the set*

$$\Sigma_a := \left\{ (z, w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z) w^j = 0 \right\}$$

lies entirely in $\overline{\mathbb{D}} \times D(0;2)$. *Let* Ω *be a connected neighbourhood of* $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$ *such that* $\Omega \cap \big(\mathbb{D} \times D(0;2)\big)$ *is connected. Then, for every* $f \in \mathcal{O}(\Omega)$, $\exists F \in \mathcal{O}(\mathbb{D} \times D(0;2))$ *such that*

$$F\Big|_{\Omega \cap \big(\mathbb{D} \times D(0;2)\big)} \equiv f\Big|_{\Omega \cap \big(\mathbb{D} \times D(0;2)\big)}.$$

Remark 7.1. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the classes of functions appearing in (7.2) and (7.3) respectively. In Chapter 6, we pointed out that the class $\mathfrak{F}_1$ is quite restrictive. While adapting the approach outlined above, we found that we could construct the deformation $\{\Phi_t : t \in [0,1]\}$ in a slightly different fashion from what is suggested in [2], which allows us to work with $a_0, ..., a_{k-1}$ belonging to a less restrictive class. Note that $\mathfrak{F}_2 \supsetneq \mathfrak{F}_1$; simply observe that if $\psi(z, w) := (M + \varepsilon)^{-1} \exp(z - w - 2)$, where $M = \sup_{(\zeta,s) \in \overline{\mathbb{D}} \times [0,1]} |\exp(\zeta - s\overline{\zeta} - 2)|$, then $M < 1$ and for $\varepsilon \in (M, 1)$, $\psi \in \mathfrak{F}_2$ but $\psi \notin \mathfrak{F}_1$.

Remark 7.2. Unbeknownst to me, Černe and Flores [4] have independently used the three-step method summarized earlier to prove:

($*$) *Let* $a_0, ..., a_{k-1}$ *be continuous functions on* $\overline{\mathbb{D}}$ *and let*

$$\Sigma_a := \{(z, w) \in \overline{\mathbb{D}} \times \mathbb{C} : w^k + a_{k-1}(z) w^{k-1} + \cdots + a_0(z) = 0\}$$

be a continuous variety over $\overline{\mathbb{D}}$. *Then, every function holomorphic in a connected neighbourhood of the set* $S = \Sigma_a \cup (\partial\mathbb{D} \times \mathbb{C})$ *extends holomorphically to a neighbourhood of* $\overline{\mathbb{D}} \times \mathbb{C}$.

Note that $\mathcal{C}(\overline{\mathbb{D}}; \mathbb{D})$ is a subset of the uniform closure (on $\overline{\mathbb{D}}$) of the function space obtained if we drop the bound "$\sup_{(z,\zeta) \in \mathbb{D}^2} |\psi(z, \zeta)| < 1$" from $\mathfrak{F}_1$. It is this, coupled with their reliance on the three-step method outlined above, that compels Černe-Flores to work with the *unbounded cylinder* $\overline{\mathbb{D}} \times \mathbb{C}$. Theorem 7.3 represents an alternative setting in which to exploit the same method with — in contrast to Černe-Flores [4] — the following initial objectives:

- to use the ideas of Barrett and Bharali to demonstrate an analytic-continuation theorem stated for a *compact* Hartogs figure ($S = \Sigma_a \cup \big(\partial\mathbb{D} \times \overline{D(0;2)}\big)$ in our case); and
- to extend the applicability of Result 7.1 to a less restrictive class of graphs/coefficients, namely $\mathfrak{F}_2$.

Due to considerations inherent to the three-step method we intend to use — see Remark 7.4(i) below — we, just like Černe-Flores, cannot work with the Hartogs configuration $\Sigma_a \cup (\partial\mathbb{D} \times \overline{\mathbb{D}})$ either. However, we *can* state a result involving $\Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$.

We refer to several lemmas from Barrett-Bharali [1] during the course of this chapter. For the sake of convenience, we first provide the statements of the relevant lemmas from [1].

Useful lemmas from Barrett-Bharali [1]

Lemma 7.4. *Let $G(re^{i\theta}) = \sum_{n=0}^{N} b_n(r)e^{in\theta}$ — i.e., we assume that $G(re^{i\cdot})$ $\forall r \in (0,1]$ has no negative Fourier modes. Assume further that $G \in \mathcal{C}^\infty(\overline{\mathbb{D}};\mathbb{C})$. Then, the holomorphic function*

$$\mathcal{D}_r(\zeta) = \sum_{n=0}^{N} b_n(r)\left(\frac{\zeta}{r}\right)^n, \quad \zeta \in D(0;r),$$

which belongs to $\mathcal{O}[D(0;r)] \cap \mathcal{C}[\overline{D(0;r)}]$, satisfies $\mathcal{D}_r(re^{i\theta}) = G(re^{i\theta})$ $\forall\theta[0,2\pi)$. Fix $\nu \in \mathbb{N}$ and let $K \Subset D(0;1-1/\nu)$ be a compact subset. The function $(r,\zeta) \mapsto \mathcal{D}_r(\zeta)$ is a continuous function on $[1-1/\nu,1] \times K$.

Lemma 7.5. *Let G be as in Lemma 7.4. Then,*

- *$\{\mathcal{D}_r\}_{r\in(0,1)}$ is a continuous family in the sense that for a fixed $\zeta_0 \in \mathbb{D}$, $r \mapsto \mathcal{D}_r(\zeta_0)$ is continuous in the interval $(|\zeta_0|,1)$.*
- *$\lim_{r\to1^-} \mathcal{D}_r(\zeta)$ exists for each $\zeta \in D$, and this limit defines a holomorphic function $\psi \in \mathcal{O}(\mathbb{D})$.*

Lemma 7.6. *Let $F \in \mathcal{C}(\overline{\mathbb{D}};\mathbb{C})$. Assume that $F(re^{i\cdot})$ $\forall r \in (0,1]$ has no negative Fourier modes. Then, given $\varepsilon > 0$ there exists a function $G \in \mathcal{C}^\infty(\overline{\mathbb{D}};\mathbb{C})$ of the form*

$$G(re^{i\theta}) = \sum_{n=0}^{N} B_n(r)e^{in\theta},$$

where N is some large positive integer and $B_n \in \mathcal{C}^\infty([0,1];\mathbb{C})$, such that $|F(\zeta) - G(\zeta)| < \varepsilon$ $\forall\zeta \in \overline{\mathbb{D}}$.

Remark. In [1], Lemma 7.6 is stated with the condition $\sup_{\partial\mathbb{D}} |F| < 1$ among the hypotheses. This condition is not needed to obtain the conclusion above, but to derive other conclusions that are needed in [1], but not in the present chapter. Thus, we have suppressed the condition $\sup_{\partial\mathbb{D}} |F| < 1$ and its associated conclusions in our version of Lemma 7.6.

Many of the mathematical details underlying Step 2 and 3 are common to Theorems 7.2 and 7.3. We now present these technicalities.

Preliminary Lemmas

The following notation will be used:

- $\mathcal{C}^\infty(\overline{\mathbb{D}};\mathbb{C})$ will denote the class of infinitely differentiable functions on the unit disc, all of whose derivatives extend to functions in $\mathcal{C}(\overline{\mathbb{D}})$;
- for $\alpha := (\alpha_0, ..., \alpha_{k-1}) \in \mathcal{C}(\overline{G};\mathbb{C}^k),\ k \in \mathbb{N}, G \subset \mathbb{C}$ a bounded domain, and $E \subset \overline{G}$

$$\mathcal{P}_\alpha(z,w) := w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j,$$

$$\Sigma_{\alpha,E} := \left\{ (z,w) \in E \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j = 0 \right\},$$

and, for the sake of convenience, the subscript E shall be dropped when $E = \overline{\mathbb{D}}$, i.e., $\Sigma_{\alpha,\overline{\mathbb{D}}} =: \Sigma_\alpha$.

The first step of the three-step strategy outlined in the first section is not difficult, but the details involved are theorem-specific. This is, in part, due to the requirements described in Step 2. The task of determining sufficient, yet not too strong, conditions on the coefficient k-tuple $(a_0, ..., a_{k-1})$ that will enable us to establish that each $\Sigma_t,\ t \in [0,1]$, is contained in the bidisc relevant to each theorem is a crucial one. The following lemma — a maximum principle for varieties — will prove useful.

Lemma 7.7. *Let $G \subset \mathbb{C}$ be a bounded domain and $a \in \mathcal{O}(G;\mathbb{C}^k) \cap \mathcal{C}(\overline{G};\mathbb{C}^k)$. Define*

$$M(z) := \max \left\{ |w| : (z,w) \in \Sigma_{a,\overline{G}} \right\}.$$

If $M(z) \leq K\ \forall z \in \partial G$, then $M(z) \leq K\ \forall z \in \overline{G}$.

Proof. We would be done if we could obtain the conclusion of this lemma when $\Sigma_{a,G}$ is an irreducible subvariety. For $\Sigma_{a,G}$ irreducible, if we can show that M is subharmonic, then the result would follow from the maximum principle.

Recall that the zeros of monic degree-k polynomials over $\mathbb{C}$, viewed as *unordered* k-tuples of zeros repeated according to multiplicity, vary continuously with the coefficients. Hence, as M is symmetric in the zeros of $\mathcal{P}_a$, $M \in \mathcal{C}(\overline{G})$.

Now, let

$$\mathfrak{R}(z) := \text{ resultant of } \mathcal{P}_a(z,\cdot) \text{ and } \partial_w \mathcal{P}_a(z,\cdot),\ z \in G.$$

By the irreducibility of $\Sigma_{a,G}$, $\mathfrak{R} \not\equiv 0$. As $\mathfrak{R} \in \mathcal{O}(G)$, $\mathfrak{S} := \mathfrak{R}^{-1}\{0\}$ is a discrete set in G. Now, for any $z_0 \in G \setminus \mathfrak{S}$, $\Sigma_{a,\{z_0\}} = \{(z_0, w_{0,1}), ..., (z_0, w_{0,k})\}$ with $w_{0,j} \neq w_{0,l}$ for $j \neq l$. As

$\partial_w \mathcal{P}_a(z_0, w_{0,j}) \neq 0$ for each $j = 1, ..., k$, we may apply the implicit function theorem at each point of $\Sigma_{a,\{z_0\}}$ to obtain a common radius $r(z_0) > 0$ such that the k sheets of $\Sigma_{a,D(z_0;r(z_0))}$ are the graphs of functions $\phi_1^{z_0}, ..., \phi_k^{z_0} \in \mathcal{O}\big(D\left(z_0; r(z_0)\right)\big)$. Clearly,

$$M(z) = \max_{j \leq k} \left|\phi_j^{z_0}(z)\right| \quad \forall z \in D\big(z_0; r(z_0)\big).$$

Thus, $M\big|_{D(z_0;r(z_0))}$ is subharmonic. As z_0 was arbitrarily chosen from the open set $G \setminus \mathfrak{S}$, we infer that $M\big|_{G\setminus\mathfrak{S}}$ is a subharmonic function.

As $\mathfrak{S}$ is the zero set of a holomorphic function, it is a polar set. But $M\big|_{G\setminus\mathfrak{S}}$ is a bounded subharmonic function, and $M \in \mathcal{C}(\overline{G})$. Therefore, M must be subharmonic in G. $\square$

Remark 7.3. The following is a paraphrasing of the above lemma that will be used in our situation.

Let $G \subset \mathbb{C}$ be a bounded domain and $a \in \mathcal{O}(G; \mathbb{C}^k) \cap \mathcal{C}(\overline{G}; \mathbb{C}^k)$. Then,

$$\Sigma_{a,\partial G} \subset \partial G \times D(0; K) \Rightarrow \Sigma_{a,\overline{G}} \subset \overline{G} \times D(0; K).$$

Remark 7.4. We will also need the following algebraic facts:

(i) If $\alpha_0, ..., \alpha_{k-1} \in \mathbb{D}$, $k \in \mathbb{N}$, and $w_1, ..., w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w^1 + \alpha_0$, then $w_j \in D(0; 2)$, $j = 1, ..., k$. For an easy proof of this fact, one can apply Rouché's theorem to $f(w) := w^k$ and $g(w) := w^k + \sum_{j=0}^{k-1} \alpha_j w^j$ on $\partial D(0; 2)$.

(ii) If $(\alpha_0, ..., \alpha_{k-1}) \in \mathbb{C}^k$, and $w_1, ..., w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w + \alpha_0$, then, for $\eta \in \mathbb{C}$, $w_1 + \eta, ..., w_k + \eta$ are the zeros of the polynomial

$$w^k + \alpha_{k-1}^{(\eta)} w^{k-1} + \cdots + \alpha_1^{(\eta)} w + \alpha_0^{(\eta)},$$

where, for each j,

$$\alpha_j^{(\eta)} = \alpha_j + \sum_{l=j+1}^{k} (-1)^{l-j} \binom{l}{l-j} \alpha_l \eta^{l-j} \tag{7.4}$$

interepreting $\alpha_k := 1$.

Theorems having a similar flavour as Theorems 7.2 and 7.3 have relied upon the Kontinuitätssatz. However, the earliest (and partially correct) works do not specify *which* form of the "Kontinuitätssatz" they rely upon. We wish, here, to make clear that the version that works for us is the version of Chirka and Stout [7]. However, merely using the Chirka-Stout Kontinuitätssatz will yield a conclusion weaker than desired, on the envelope of holomorphy of the domain in question. The next lemma follows the approach of Barrett and Bharali [1] to argue that it is, in fact, possible to obtain the strong conclusion of Chirka's extension theorem (i.e. Theorem 4.1) in our situation.

Lemma 7.8. *Let* $a = (a_0, ..., a_{k-1}) \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ *and* $\Sigma_a \subset \overline{\mathbb{D}} \times D(0;r)$, $r > 0$. *Let* Ω *be a connected open neighbourhood of* $S := \Sigma_a \cup \big(\partial\mathbb{D} \times \overline{D(0;r)}\big)$ *and* $f \in \mathcal{O}(\Omega)$. *Let* $V := Ann(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, *be such that* $V \times D(0;r) \subset \Omega$, *and let* $D \Subset \Omega$, *be an open subset containing* S. *For any* $\alpha \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ *and any* $\eta \in \mathbb{C}$, *let* $\alpha^{(\eta)} \in \mathcal{C}(\overline{\mathbb{D}}; \mathbb{C}^k)$ *denote the perturbation that is given by (7.4) so that* $\Sigma_{\alpha^{(\eta)}} = \Sigma_\alpha + (0, \eta)$. *Suppose there exists a continuous function* $A := (A_0, ..., A_{k-1}) : \overline{\mathbb{D}} \times [0,1]$, *and a* $\delta > 0$ *which is so small that, defining* $\Sigma_t^\eta := \Sigma_{A^{(\eta)}(\cdot,t)}$, *we have*

- *(i) for each* $\eta \in D(0;\delta)$, $\Sigma_t^\eta \subset \overline{\mathbb{D}} \times D(0;r)$ $\forall t \in [0,1]$; *and*
- *(ii) for each* $\eta \in D(0;\delta)$, $\Sigma_t^\eta \cap (\mathbb{D} \times D(0;r)) \setminus \overline{D}$ *is a complex-analytic subvariety of* $\mathbb{D} \times D(0;r) \setminus \overline{D}$.

Then, there exists a connected neighbourhood Ω_1 *of* $S_1 := \Sigma_1^0 \cup \big(\partial\mathbb{D} \times D(0;r)\big)$ *and* $f_1 \in \mathcal{O}(\Omega_1)$ *such that*

$$f_1\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)}.$$

Proof. Let

$$\mathcal{T} := \bigcup_{\eta \in D(0;\delta)} \Sigma_1^0 + (0, \eta).$$

By the Chirka-Stout Kontinuitätssatz [7], $\mathcal{T} \subset \pi(\widetilde{\Omega})$, where $(\widetilde{\Omega}, \pi)$ denotes the envelope of holomorphy of Ω.

There is a canonical holomorphic imbedding of Ω into $\widetilde{\Omega}$. We denote this imbedding by $j : \Omega \hookrightarrow \widetilde{\Omega}$. Corresponding to each $f \in \mathcal{O}(\Omega)$, there is a holomorphic function $\mathcal{E}(f) \in \mathcal{O}(\widetilde{\Omega})$ such that $\mathcal{E}(f) \circ j = f$. By [7] (and analogous to the situation in [1]), there exists a holomorphic mapping (note that Σ_1^η varies analytically in η) $H : \mathcal{T} \to \widetilde{\Omega}$ such that

$$\pi \circ H\big(\Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w)\big) = \Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w) \quad \forall \eta \in D(0;\delta) \text{ and } z \in \overline{\mathbb{D}}.$$

Now, for each $p := (z_1, w_1) \in \mathcal{T} \cap \big(V \times D(0;r)\big)$, there exist

- an $\eta_0 \in D(0;\delta)$; and
- a point $q \in \Sigma_0^{\eta_0} \cap \{z_1\} \times \mathbb{C}_w$,

such that the continuous family $\{\Sigma_t^{\eta_0}\}_{t\in[0,1]}$ determines a path $\gamma_{qp} : [0,1] \to \{z_1\} \times \mathbb{C}_w$ with $\gamma_{qp}(0) = q$ and $\gamma_{qp}(1) = p$. Let $\mathfrak{S}_\Omega :=$ the sheaf of $\mathcal{O}(\Omega)$-germs over $\mathbb{C}^2$ (refer to [13, Chapter 6] for the definition of an $\mathcal{O}(\Omega)$-germ) and let

$$\widetilde{\gamma_{qp}} := \text{the lift of } \gamma_{qp} \text{ to } \mathfrak{S}_\Omega \text{ starting at the germ } [g : g \in \mathcal{O}(\Omega)]_q.$$

Examining the Kontinuitätssatz, $H(p) = \widetilde{\gamma_{qp}}(1)$.

We know that if $[s_g : g \in \mathcal{O}(\Omega)]_z$ is an $\mathcal{O}(\Omega)$-germ in $\widetilde{\Omega}$, then

$$\mathcal{E}(f)\left([s_g : g \in \mathcal{O}(\Omega)]_z\right) = s_f(z).$$

By the monodromy theorem, $\widetilde{\gamma_{qp}}(1) = [g : g \in \mathcal{O}(\Omega)]_p$. Hence,

$$\mathcal{E}(f) \circ H(p) = \mathcal{E}(f)\left(\widetilde{\gamma_{qp}}(1)\right) = f(p).$$

Since the above holds for any arbitrary $p \in \mathcal{T} \cap \big(V \times\ D(0;r)\big)$, we see that

$$\mathcal{E}(f) \circ H = f \text{ on } \mathcal{T} \cap \big(V \times\ D(0;r)\big).$$

Finally, let $\Omega_1 := \mathcal{T} \cup (V \times D(0;r))$ and

$$f_1(z,w) := \begin{cases} \mathcal{E}(f) \circ H(z,w), & \text{if } (z,w) \in \mathcal{T}, \\ f(z,w), & \text{if } (z,w) \in V \times D(0;r), \end{cases}$$

Then, $f_1 \in \mathcal{O}(\Omega_1)$ and

$$f_1\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times D(0;r)\right)}.$$

□

Proof of Theorem 7.2

By Lemma 7.6 and the continuous dependence of the zeros of a polynomial on its coefficients, we know that it is enough to prove Theorem 7.2 for $a_0, ..., a_{k-1} \in \mathfrak{G}_1$, where $\mathfrak{G}_1 \subsetneq \mathcal{C}(\overline{\mathbb{D}};\mathbb{C})$ is the following set:

$$\left\{ g \in \mathcal{C}^\infty(\overline{\mathbb{D}};\mathbb{C}) : \exists N \in \mathbb{N},\ G_n \in \mathcal{C}^\infty\big([0,1];\mathbb{C}\big) \text{ such that } g(re^{i\theta}) = \sum_{n=0}^{N} G_n(r)e^{in\theta},\ r \in (0,1] \right\}.$$

Thus, we replace $a = (a_0, ..., a_{k-1})$ in Theorem 7.2 by $b := (b_0, ..., b_{k-1}) \in \mathfrak{G}_1^k$. This is because we can find a Σ_b that is so close to Σ_a that $\Sigma_b \subset \Omega$ and is attached to $\partial\mathbb{D} \times \overline{\mathbb{D}}$.

Fix a $j \in \{0, ..., k-1\}$. Let

$$b_j(re^{i\theta}) = \sum_{n=0}^{n(j)} B_n^j(r)e^{in\theta}, \ \theta \in [0, 2\pi),$$

where $n(j) \in \mathbb{N}$ and $B_n^j \in \mathcal{C}^\infty\big([0,1];\mathbb{C}\big)$. Using Lemma 7.4 in [1], where Barrett and Bharali constructed an explicit family of analytic discs in $\overline{\mathbb{D}}\times\mathbb{C}$ with boundaries in $\{(z, b_0(z), ..., b_{k-1}(z)) : z \in \mathbb{D}\}$, we define a family of continuous discs $\{\mathfrak{B}_t = (\mathfrak{B}_{t,0}, ..., \mathfrak{B}_{t,k-1})\}_{t\in[0,1]}$ as follows:

$$\mathfrak{B}_{t,j}(\zeta) := \begin{cases} \sum_{n=0}^{n(j)} B_n^j(t)\left(\frac{\zeta}{t}\right)^n, & \text{if } \zeta \in D(0;t), \\ \\ b_j(\zeta), & \text{if } \zeta \in \overline{Ann(0;t,1)}. \end{cases} \tag{7.5}$$

Note that $\mathfrak{B}_0 = b$. Also, by Lemma 7.5 in [1], $\{\mathfrak{B}_t\}_{t\in[0,1]}$ is a continuous family, and $\mathfrak{B}_1 \in \mathcal{O}(\mathbb{D};\mathbb{C}^k) \cap \mathcal{C}(\overline{\mathbb{D}};\mathbb{C}^k)$.

Let $\delta > 0$ be so small that $\eta \in D(0;\delta) \Rightarrow \Sigma_b + (0,\eta) \subset \Omega \cap \big(\overline{\mathbb{D}} \times \mathbb{D}\big)$. Let $b^{(\eta)} = \left(b_1^{(\eta)}, ..., b_{k-1}^{(\eta)}\right)$ be defined pointwise by (7.4) in the previous section. By Remark 7.4(ii), each $b_j^{(\eta)}$, being a linear combination of $b_j, ..., b_{k-1}$, is in $\mathfrak{G}_1$. Thus, we can define continuous discs $\left\{\mathfrak{B}_t^{(\eta)} = \left(\mathfrak{B}_{t,0}^{(\eta)}, ..., \mathfrak{B}_{t,k-1}^{(\eta)}\right)\right\}_{t\in[0,1]}$ using the Fourier coefficients of $b_j^{(\eta)}(re^{i\cdot})$, $r \in (0,1]$, just as in equation (7.5). It is a simple observation that the same discs can be obtained by defining, on $\overline{\mathbb{D}}$,

$$\mathfrak{B}_{t,j}^{(\eta)} := \mathfrak{B}_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j}\binom{l}{l-j}\mathfrak{B}_{t,l}\eta^{l-j} + (-1)^{k-j}\binom{k}{k-j}\eta^{k-j}. \tag{7.6}$$

It is important to note that $\mathfrak{B}_t^{(0)} \equiv \mathfrak{B}_t \ \forall t \in [0,1]$.

Fix a domain $D \Subset \Omega$, such that $S \subset D$. We claim that the continuous family $\left\{\mathfrak{B}_t^{(\eta)}\right\}_{t\in[0,1]}$ satisfies the following properties:

a) $\mathfrak{B}_0^{(\eta)} = b^{(\eta)} \ \forall \eta \in D(0;\delta)$;

b) for a fixed t, $\mathfrak{B}_t^{(\eta)}$ depends analytically on η;

c) for each $\mathfrak{B}_t^{(\eta)}$, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \setminus \overline{D}$ is an analytic subvariety of $\overline{\mathbb{D}} \times \mathbb{C} \setminus \overline{D}$; and

d) for each t, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \subset \overline{\mathbb{D}} \times \mathbb{D} \ \forall \eta \in D(0;\delta)$.

Properties a) and b) follow from construction. For c), it is enough to observe that

$$\Sigma_{\mathfrak{B}_t^{(\eta)}} = \left(\Sigma_{b^{(\eta)},\overline{Ann(0;t,1)}}\right) \bigcup \left(\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)}\right),$$

and that $\mathfrak{B}_t^{(\eta)}\big|_{D(0;t)} \in \mathcal{O}(D(0;t);\mathbb{C}^k)$. For property d), it is enough to show that

$$\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)} \subset \overline{\mathbb{D}} \times \mathbb{D}.$$

But this follows from Lemma 7.7 applied to $\Sigma_{\mathfrak{B}_t^{(\eta)},D(0;t)}$, with $D(0;t)$ acting as G, since

$$\Sigma_{\mathfrak{B}_t^{(\eta)},\partial D(0;t)} \equiv \Sigma_{b^{(\eta)},\partial D(0;t)} \subset \partial D(0;t) \times \mathbb{D}.$$

From this, we can conclude that the mapping $A : \overline{\mathbb{D}} \times [0,1] \to \mathbb{C}^k$ with $A(z,t) := \mathfrak{B}_t(z)$ satisfies the hypotheses of Lemma 7.8. Thus, there exists a connected open neighbourhood Ω_1 of $S_1 := \Sigma_{\mathfrak{B}_1^{(0)}} \cup \left(\partial\mathbb{D} \times \mathbb{D}\right)$ and a $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1\big|_{\Omega_1 \cap \left(V \times \mathbb{D}\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times \mathbb{D}\right)},$$

where $V := Ann(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, such that $V \times \mathbb{D} \subset \Omega$.

But, $\mathfrak{B}_1^{(0)}$ is holomorphic by construction. Hence, from the analytic-multigraph version of Hartogs' lemma, $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega_1 \cap \mathbb{D}^2} \equiv f_1\big|_{\Omega_1 \cap \mathbb{D}^2}.$$

Thus, F and f must coincide in $\Omega_1 \cap \left(V \times \mathbb{D}\right) \cap \mathbb{D}^2$. As the latter is an open subset of the connected set $\Omega \cap \mathbb{D}^2$,we conclude that

$$F\big|_{\Omega \cap \mathbb{D}^2} \equiv f\big|_{\Omega \cap \mathbb{D}^2}.$$

□

Proof of Theorem 7.3

The proof of this theorem is similar to that of Theorem 7.2. The main difference lies in the specific method of constructing, starting from the given multigraph, a continuous family of multigraphs along which we can achieve analytic continuation by invoking the Kontinuitätssatz. Recall that, in the previous section, the form of each coefficient function a_j facilitated the construction of functions that were holomorphic on increasing concentric discs in $\mathbb{D}$. In the present case, to perturb the coefficients, we will construct analytic annuli attached to the graphs of a_j along their inner boundaries, and to $\partial\mathbb{D}\times\mathbb{D}$ along their outer boundaries. In view of Remark 7.4(i), we are compelled to work with a polydisc longer than $\mathbb{D}^2$.

Proof of Theorem 7.2. Let $a(z) = \psi(z,\overline{z}) := (\psi_0(z,\overline{z}), ..., \psi_{k-1}(z,\overline{z}))$. Set $\mathcal{R} := \overline{\mathbb{D}}\times[0,1]$. Note that, by hypothesis, we can find an $\varepsilon > 0$ such that $Ann(0; 1-2\varepsilon, 1+2\varepsilon)\times\mathbb{D}(0;2)\subset\Omega$. Hence, just as in the proof of Theorem 6.1 in Chapter 6, it suffices to work with $\Sigma_{a,\overline{D(0;1-\varepsilon)}}$ and the Hartogs configuration $S_\varepsilon := \Sigma_{a,\overline{D(0;1-\varepsilon)}}\cup\big(\partial D(0;1-\varepsilon)\times\overline{D(0;2)}\big)$. This affords us the very useful property:

$$(\zeta, s)\mapsto\psi_j(\zeta, s\overline{\zeta}) \text{ is continuous on } \overline{D(0;1-\varepsilon)}\times[0,1],\ \forall j = 0, ..., k-1.$$

Therefore, it actually suffices to prove Theorem 7.3 under the assumption that $\psi_0, ..., \psi_{k-1}\in\mathfrak{G}_2$, where

$$\mathfrak{G}_2 := \left\{\psi\in\mathcal{O}(\mathbb{D}^2)\cap\mathcal{C}(\overline{\mathbb{D}}^2) : \sup_{(\zeta,s)\in\mathcal{R}}|\psi(\zeta, s\overline{\zeta})| < 1\right\}.$$

In order to avoid messy subscripted notation such as $\Sigma_{a,\overline{D(0;1-\varepsilon)}}$ and messy normalizations, we shall hereafter assume that $\psi_j\in\mathfrak{G}_2$, for $j = 0, ..., k-1$.

We define a family of continuous discs $\{\Psi_t = (\Psi_{t,0}, ..., \Psi_{t,k-1})\}_{t\in[0,1)}$ as follows:

$$\Psi_t(\zeta) := \begin{cases} a(\zeta) = \psi(\zeta,\overline{\zeta}), & \text{if } \zeta\in D(0;1-t), \\ \\ \psi\left(\zeta, \frac{(1-t)^2}{\zeta}\right), & \text{if } \zeta\in\overline{Ann(0;1-t,1)}. \end{cases} \tag{7.7}$$

Therefore, $\Psi_0 = a$. We observe that $\{\Psi_t\}_{t\in[0,1)}$ is a continuous family in the sense that for a fixed $\zeta_0\in\overline{\mathbb{D}}$, $t\mapsto\Psi_t(\zeta_0)$ is continuous in the interval $[0,1)$. Furthermore, we may define

$$\Psi_1(\zeta) := \lim_{t\to 1^-}\Psi_t(\zeta) = \psi(\zeta, 0), \tag{7.8}$$

Thus, $\Psi_1\in\mathcal{O}(\mathbb{D};\mathbb{C}^k)$. Also, note that, for each $t\in[0,1]$,

$$\sup_{\zeta\in\partial\mathbb{D}}|\Psi_{t,j}(\zeta)| = \sup_{\zeta\in\partial\mathbb{D}}\left|\psi_j(\zeta, (1-t)^2\overline{\zeta})\right| < 1,\ j = 0, ..., k-1. \tag{7.9}$$

Let $\delta > 0$ be so small that

- ► $\eta\in D(0;\delta)\Rightarrow\Sigma_a + (0,\eta)\subset\Omega\cap\big(\overline{\mathbb{D}}\times\mathbb{D}\big)$; and
- ► for all $\eta\in D(0;\delta)$ and $j = 0, ..., k-1$,

$$\sup_{(\zeta,s)\in\mathcal{R}}\left|\psi_j(\zeta, s\overline{\zeta})\right| + \sum_{l=j+1}^{k-1}\binom{l}{l-j}\sup_{(\zeta,s)\in\mathcal{R}}\left|\psi_l(\zeta, s\overline{\zeta})\right||\eta|^{l-j} + \binom{k}{k-j}|\eta|^{k-j} < 1. \tag{7.10}$$

Let $\psi^{(\eta)} = \left(\psi_1^{(\eta)}, ..., \psi_{k-1}^{(\eta)}\right) \in \mathcal{O}(\mathbb{D}^2; \mathbb{C}^k)$ be defined pointwise on $\overline{\mathbb{D}}^2$ by (7.4). By (7.10),

$$\sup_{(\zeta,s)\in\mathcal{R}} \left|\psi_j^{(\eta)}(\zeta, s\overline{\zeta})\right| < 1 \ \forall\eta \in D(0;\delta) \text{ and } j = 0, ..., k-1.$$

Thus, each $\psi_j^{(\eta)} \in \mathfrak{G}_2$.

Now, just as in the proof of Theorem 7.2, we use $\{\Psi_t\}_{t\in[0,1]}$ to construct continuous families of continuous discs $\left\{\Psi_t^{(\eta)} = \left(\Psi_{t,0}^{(\eta)}, ..., \Psi_{t,k-1}^{(\eta)}\right)\right\}_{t\in[0,1]}$, on $\overline{\mathbb{D}}$, as follows:

$$\Psi_{t,j}^{(\eta)} := \Psi_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j} \binom{l}{l-j} \Psi_{t,l}\eta^{l-j} + (-1)^{k-j} \binom{k}{k-j} \eta^{k-j}. \tag{7.11}$$

Note that $\Psi_t^{(0)} = \Psi_t$, and by construction

$$\sup_{\zeta\in\partial\mathbb{D}} \left|\Psi_t^{(\eta)}(\zeta)\right| = \sup_{\zeta\in\partial\mathbb{D}} \left|\psi_t^{(\eta)}(\zeta, (1-t)^2\overline{\zeta})\right| < 1. \tag{7.12}$$

As before, fixing a domain $D \Subset \Omega$ such that $S \subset D$, we claim that the following properties are satisfied:

a*) $\Psi_0^{(\eta)} = a^{(\eta)} \ \forall\eta \in D(0;\delta)$;

b*) for a fixed t, $\Psi_t^{(\eta)}$, depends analytically on η;

c*) for each $\Psi_t^{(\eta)}$, $\Sigma_{\Psi_t^{(\eta)}} \setminus \overline{D}$ is an analytic subvariety of $\overline{\mathbb{D}} \times \mathbb{C} \setminus \overline{D}$; and

d*) for each t, $\Sigma_{\Psi_t^{(\eta)}} \subset \overline{\mathbb{D}} \times D(0;2) \ \forall\eta \in D(0;\delta)$.

Properties a*) and b*) pose no problem, and c*) can be argued in exactly the same way as in the previous section. For *d**), we write, in the notation established in the beginning of this chapter:

$$\Sigma_{\Psi_t^{(\eta)},\partial Ann(0;1-t,1)} = \Sigma_{\Psi_t^{(\eta)},\partial D(0;1-t)} \bigcup \Sigma_{\Psi_t^{(\eta)},\partial\mathbb{D}}. \tag{7.13}$$

Note that $\Sigma_{\Psi_t^{(\eta)},\partial D(0;1-t)} \subset \partial D(0;1-t) \times D(0;2)$, while due to inequality (7.12) and Remark 7.4(i), we have that $\Sigma_{\Psi_t^{(\eta)},\partial\mathbb{D}} \subset \partial\mathbb{D} \times D(0;2)$. Thus, applying Lemma 7.7 (specifically, its paraphrasing in Remark 7.3) to $\Sigma_{\Psi_t^{(\eta)},Ann(0;1-t,1)}$, we have that *d**) holds.

From this, we conclude that the mapping $A : \overline{\mathbb{D}} \times [0,1] \to \mathbb{C}^k$ defined as $A(z,t) := \Psi_t(z)$ satisfies the hypotheses of Lemma 7.8. Thus, there exists a connected open neighbourhood

Chapter 7: Extension Theorems of Hartogs-Chirka Type

Ω_1 of $S_1 := \Sigma_{\Psi_1^{(0)}} \cup \big(\partial\mathbb{D} \times D(0;2)\big)$ and a $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1\big|_{\Omega_1 \cap \left(V \times D(0;2)\right)} \equiv f\big|_{\Omega_1 \cap \left(V \times D(0;2)\right)},$$

where $V := Ann(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, such that $V \times D(0;2) \subset \Omega$. $\Psi_1^{(0)}$ being holomorphic by construction, we can repeat the argument presented in the proof of Theorem 7.2 to conclude that $\exists F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F\big|_{\Omega\cap\mathbb{D}^2} \equiv f\big|_{\Omega\cap\mathbb{D}^2}.$$

$\square$

References

[1] D. Barrett and G. Bharali, *The role of Fourier modes in extension theorems of Hartogs-Chirka type*, Math. Z. **249** (2005), 883-901.

[2] G. Bharali, *Some generalizations of Chirka's extension theorem*, Proc. Amer. Math. Soc. **129** (2001), 3665-3669.

[3] H. Cartan and P. Thullen, *Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen*, Math. Ann. **106** (1932), no. 1, 617–647.

[4] M. Černe and M. Flores, *Some Remarks on Hartogs' Extension Lemma*, to appear in Proc. Amer. Math. Soc.

[5] E. M. Chirka, *The generalized Hartogs lemma and the non-linear $\overline{\partial}$-equation*, Complex Analysis in Modern Mathematics, FAZIS, Moscow, 2001, 19-31.

[6] E. M. Chirka and J.-P. Rosay, *Remarks on the proof of a generalised Hartogs lemma,* Ann. Pol. Math. **70** (1998), 43-47.

[7] E. M. Chirka and E. L. Stout, *A Kontinuitätssatz,* Topics in complex analysis (Warsaw, 1992), Banach Center Publ., **31**, Polish Acad. Sci., Warsaw, 1995, 143–150.

[8] P. Gupta, *Two extension theorems of Hartogs-Chirka type involving continuous multi-function*, Submittef.

[9] F. Hartogs, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselber durch Reihen welche nach Potentzen einer Veränderlichen fortschreiten,* Math. Ann. **62** (1906), 1-88.

[10] M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, 33, Springer-Verlag, 1976.

[11] J. Merker and E. Porten, *A Morse-Theoretical Proof of the Hartogs Extension Theorem*, J. Geom. Anal. **17**(3) (2007), 513-546.

[12] J. Milnor, *Morse Theore*, Based on lecture notes by M. Spivak and R. Wells, Annals of Mathematical Studies, No. 51, Princeton University Press, 1963.

[13] R. Narasimhan, *Several Complex Variables,* University of Chicago Press, Chicago, 1971.

[14] J.-P. Rosay, *A counterexample related to Hartogs' phenomenon (A question by E. Chirka)*, Michigan Math. J. **45** (1998), 529-535.