Frustration induced oscillator death on networks

Prashant M. Gade[1] and Govindan Rangarajan[2]
[1]PG Department of Physics, Rashtrasant Tukdoji Maharaj Nagpur University, Nagpur 440033, India
[2]Department of Mathematics and Centre for Neuroscience, Indian Institute of Science, Bangalore 560 012, India

(Received 11 April 2013; accepted 19 June 2013; published online 9 July 2013)

An array of identical maps with Ising symmetry, with both positive and negative couplings, is studied. We divide the maps into two groups, with positive intra-group couplings and negative inter-group couplings. This leads to antisynchronization between the two groups which have the same stability properties as the synchronized state. Introducing a certain degree of randomness in signs of these couplings destabilizes the anti-synchronized state. Further increasing the randomness in signs of these couplings leads to oscillator death. This is essentially a frustration induced phenomenon. We explain the observed results using the theory of random matrices with nonzero mean. We briefly discuss applications to coupled differential equations. © *2013 AIP Publishing LLC*.
[http://dx.doi.org/10.1063/1.4812797]

We study coupled maps on a random network. When we introduce positive as well as negative couplings on such a network, geometric frustration is introduced, leading to reduction of eigenvalues and oscillator death. We give an analytic explanation of these results using theory of random matrices with nonzero mean.

I. INTRODUCTION

Amplitude/oscillator death is a phenomenon in which two or more autonomously oscillating systems approach a stable zero-amplitude state when coupled.[1] Amplitude death in coupled oscillators has been studied in various contexts and several routes to amplitude death have been reported.[1] Here, we report another route to amplitude death in coupled identical oscillators on a network. The network has both positive and negative couplings.[2] Presence of couplings of both signs induces frustration and leads to amplitude death. Though not studied extensively, it is known that negative and positive couplings could co-exist in several networks. In neurons, both excitatory and inhibitory synaptic inputs co-exist while in ecological webs interactions between species could have both signs.[3,4] They are also observed in laboratory systems such as coupled lasers.[5] Equations with negative Laplacian appear in the context of several pattern forming systems. Examples of this type are Kuramoto-Sivashinsky and Swift-Hohenberg equations.[6] With negative coupling, anti-synchronization can occur as well. We study both anti-synchronization and amplitude death in this system.

We divide the oscillators into two groups. First, we choose positive intra-group and negative inter-group couplings. This leads to an antisynchronized state with the same stability properties as the synchronized state for all positive couplings. These states are shown to have the same stability properties as the synchronized state obtained for all negative couplings or anti-synchronized state for negative intra-group and positive inter-group couplings. If we relax the condition of having uniform signs for intra-group or inter-group couplings, the synchronized or anti-synchronized state does not exist. Given enough randomness in the connections, all the oscillators approach 0 state, i.e., amplitude death. We explain these observations by carrying out a stability analysis of this state. This work is perhaps the first application of the recently developed theory of random matrices with nonzero mean.

The above phenomena are not merely of theoretical interest. There are various examples where the system is seen to be divided into groups and couplings within a group are different from couplings outside the group. For example, in magnetic systems there are layered systems called metamagnets in which couplings within a layer are ferromagnetic and those between layers are anti-ferromagnetic.[7] Such connections are also found in social networks.[8]

The plan of the paper is as follows. In Sec. II, we define the model and demonstrate the equivalence between the state of synchronization of all elements and that of anti-synchronization between two groups. In Sec. III, we introduce a specific random-neighbor model which is studied in detail. We show that frustration in connections indeed leads to amplitude death in coupled maps and explain these results analytically. In Sec. IV, we study coupled Lorenz systems on such networks. In Sec. V, we present our conclusions and possible directions for further research.

II. MODEL

Our model is that of a coupled map lattice on an arbitrary network of n elements

$$x_i(t+1) = \sum_{j=1}^{n} A_{ij} f(x_j(t)), \tag{1}$$

where A_{ij} is (i, j)th element of the adjacency/connectivity matrix A of the underlying network. The indices i and j take values from 1 to n. We focus on systems with Ising symmetry where the function $f(x)$ has an antisymmetric form

$f(-x) = -f(x)$. (For a similar usage, see Miller and Huse.[9]) In particular, we choose $f(x) = \alpha x^3 + (1-\alpha)x$.

First, we consider a system without frustration. The n elements are decomposed into two groups U (comprising the first m elements) and V (comprising the next $n-m$ elements). Let all the connections be positive. Let the connectivity matrix be row stochastic, that is $\sum_{j=1}^{n} A_{ij} = r$ for all i, so that synchronized state can exist. The connectivity matrix can be written in block form as

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{2}$$

where P and S are intra-group connections, while Q and R are inter-group connections. Stability of synchronized state can be inferred from the Jacobian. The elements of Jacobian are given by $J_{ij}(t) = \partial x_i(t+1)/\partial x_j(t) = A_{ij} f'(x_j(t))$ When $x_j(t) = x^*(t)$ for all j, $J_{ij}(t) = A_{ij} f'(x^*(t))$ and hence $J(t) = Af'(x^*(t))$. Thus, the Jacobian for synchronous state is the connectivity matrix A multiplied by $f'(x^*(t))$.

Now, if we change the signs of all inter-group couplings we get a connectivity matrix A' where

$$A' = \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}. \tag{3}$$

We can have an antisynchronized state for this network. The elements of Jacobian for this state are given by $J_{ij} = A'_{ij} f'(x^*(t))$ or $A'_{ij} f'(-x^*(t))$ depending on the group the site j belongs to. However, since our map has odd parity, its derivative has even parity and $f'(-x^*(t)) = f'(x^*(t))$. Thus $J = A' f'(x^*(t))$ for the antisynchronized state.

Define a matrix

$$D = \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}, \tag{4}$$

where I_m and I_{n-m} are identity matrices in m and $n-m$ dimensions. It is easily seen that $A' = D^{-1}AD$. Therefore, matrices A and A' are related by a similarity transformation and hence have the same eigenvalues. As we have seen above, the Jacobian for the synchronized or anti-synchronized state is simply the appropriate connectivity matrix (A or A', respectively) multiplied by $f'(x^*(t))$. Since A and A' are related by a similarity transformation, the stability of synchronized state with all positive couplings is the same as stability of the anti-synchronized state with positive intra-group and negative inter-group couplings.

What happens if we reverse signs of all the couplings of matrix A? Eigenvalues of the connectivity matrix change their signs, but not their magnitudes. Stability properties, since they depend only on eigenvalue magnitudes, remain unchanged. The synchronized state is still a possible state since the matrix remains row stochastic. However, all variables change signs at all odd times. We also note that matrix $-A$ is related to $-A'$ by the same similarity transformation as above. As noted above, this transformation changes signs of the elements of the second group. Hence, stability of the synchronized state with all negative connections is the same as that of the antisynchronized state with negative intra-group and positive inter-group connections.

In summary, synchronized state with all positive (or negative) connections and anti-synchronized state with positive (negative) inter-group and negative (positive) intra-group connections have the same stability properties.

In Sec. III, we address the following question: What happens if there is inconsistency in the signs of connections. First, the synchronized or antisynchronized state may not even exist. For example, the connectivity matrix should be row stochastic for synchronized state and this will not be the case any longer. However, the amplitude death state still exists. This is a synchronized state and its stability is determined by eigenvalues of A multiplied by $f'(0)$ for any coupling scheme. This is because the elements of the Jacobian are given by $J_{ij}(t) = \partial x_i(t+1)/\partial x_j(t) = A_{ij} f'(x_j(t))$. When $x_j(t) = 0$ for all j (amplitude death state), $J_{ij} = A_{ij} f'(0)$ and hence $J = Af'(0)$.

III. RANDOM NEIGHBOR MODEL

We now investigate in some detail amplitude death in the presence of inconsistency in the signs of connections for a specific topology, namely a random-network type topology, although similar results would hold for other networks. Let matrix A be of the following type. Each site i is connected with k randomly chosen neighbors. The value of $A_{i,j}$ is randomly chosen as $1/k$ or $-1/k$ for connected sites and is 0 if the sites are not connected. This model is similar to Erdös-Rényi type network in which every site is connected to exactly k randomly chosen sites.[10,11]

The above connectivity scheme leads to synchronized chaos for all positive connections since this is a row stochastic matrix.[12] We separate the sites in two groups with m and $n-m$ elements and impose a condition that inter-group connections be negative ($-1/k$) and intra-group connections be positive ($1/k$). (We have taken $m = n/2$ in simulations.) This leads to antisynchronized chaos under the same conditions.

Now, we consider the following case. Let the intra-group couplings be positive with probability p_1 and inter-group couplings be negative with probability p_2. As explained above, for $p_1 = p_2 = 1$ (or $p_1 = p_2 = 0$) we see an antisynchronized state. For $p_1 = 1, p_2 = 0$ or $p_2 = 1, p_1 = 0$, we obtain a synchronized state. What happens when we depart from these extreme conditions? When the couplings are not consistent, some intra-group connections will be negative and some inter-group connections will be positive. Hence, frustration is induced and antisynchronization is reduced. However, around $p_1 = p_2 = 1/2$ there is a big area where the oscillators are again synchronized (as well as antisynchronized). This is a state of amplitude death where $x_i(t) = 0 \forall i$.

What are conditions under which the amplitude death state can be observed? We consider the system of coupled antisymmetric cubic maps introduced earlier (cf. Eq. (1)). For the amplitude death state to be stable we need all the eigenvalues of the Jacobian matrix (evaluated at the amplitude death state) to be inside the unit circle in the complex plane. For our system, this means that all the eigenvalues of

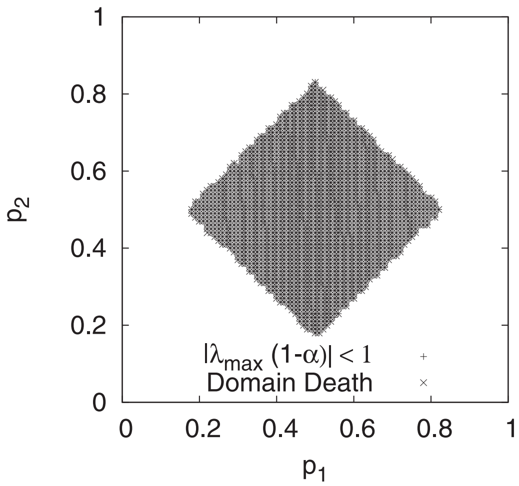

FIG. 1. Domain of oscillator death for $k = 49$, $\alpha = 3.999$ in $p_1 - p_2$ plane (marked by crosses). We also plot values of $p_1 - p_2$ for which $\lambda_{max}|f'(0)| = \lambda_{max}|1 - A| < 1$ (marked by + symbol). Both domains match as expected.

$Af'(0)$ are less than 1 in magnitude. Equivalently, denoting the maximum eigenvalue of A by λ_{max}, we have $|\lambda_{max}| < |1/f'(0)|$. For $f(x) = \alpha x^3 + (1-\alpha)x$, $f'(0) = (1-\alpha) = -2.999$ when $\alpha = 3.999$. In Fig. 1, in $p_1 - p_2$ space, we have plotted values of p_1 and p_2 for which this condition holds. (We have taken $n = 200$ and $k = 49$.) We have also plotted the numerical values of p_1 and p_2 for which amplitude death is observed. There is an excellent match between the two. Domain of oscillator death in $p_1 - p_2$ plane is diamond shaped. The borders are lines of type $p_1 - p_2 = constant$ (parallel to $p_1 = p_2$) and of type $p_1 + p_2 = constant$. It is clear that the domain of oscillator death is centered around the point of maximum frustration, i.e., $p_1 = p_2 = 1/2$.

In Fig. 2, we have plotted the absolute value of largest eigenvalue of the adjacency matrix A. It again shows an interesting symmetry. It has the shape of an inverted pyramid cut at the bottom. It is possible to explain this shape using arguments given later which also lead to an *Ansatz* for the largest eigenvalue of the adjacency matrix. Using the same logic as in Sec. II, we argue that the points (p_1, p_2), $(1-p_1, 1-p_2)$, $(1-p_1, p_2)$ and $(p_1, 1-p_2)$ have the same stability properties. This explains the 4-fold symmetry in $p_1 - p_2$ plane. The fact that the domain is bounded by lines with constant $p_1 + p_2$ or $p_1 - p_2$ demonstrates that the largest eigenvalue is dependent solely on the number of "wrong"

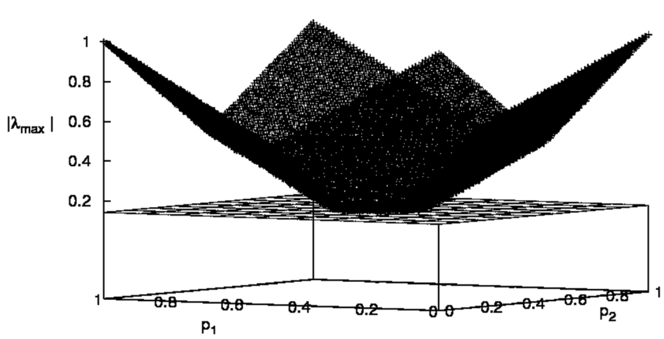

FIG. 2. We plot $|\lambda max|$ for the adjacency matrix in $p_1 - p_2$ plane A $|\lambda_{max}| = 1/\sqrt{k}$ plane demonstrates that eigenvalues are bounded from below by $1/\sqrt{k}$. The symmetry is evident.

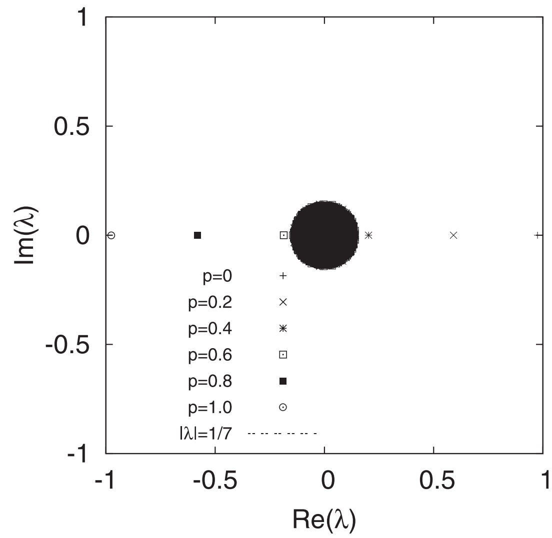

FIG. 3. Eigenvalues of the adjacency matrix are plotted in complex plane for $n = 1000$ and $k = 49$ for various values of p. We see that $n - 1$ eigenvalues are in a circle of radius 1/7 in complex plane. Circle of radius 1/7 is drawn for reference. There is also an outlier at $1 - 2p$.

connections compared to the nearest vertex. For example, for $p_1 < 1//2, p_2 < 1/2$, the number of wrong connections (with respect to (0,0) would be $(p_1 + p_2)/2$ and for $p_1 > 1/2, p_2 < 1/2$, the number of wrong connections (with respect to (0,1)) would be $(p_1 + (1 - p_2))/2$.

To simplify matters, let us construct an adjacency matrix with k nonzero connections in each row. The value of entry is $-1/k$ with probability p and $1/k$ with probability $1 - p$. Fig. 3 shows the eigenspectrum of such a matrix for $k = 49$, $n = 1000$ at various values of p. We see a circle of radius $1/\sqrt{49} = 1/7$ in the complex plane and an outlier eigenvalue. This outlier eigenvalue changes from 1 to -1 as we change p from $p = 0$ to $p = 1$. It can be well fitted by $(1 - 2p)$ (when $1 - 2p > 1/\sqrt{k}$). In Fig. 4, $|\lambda_{max}|$ is plotted a function of p. It can be fitted by $(1 - 2p)$ as long as $(1 - 2p) > 1/\sqrt{k}$ and the eigenvalue is given by $\sqrt{k}$ near $p = 1/2$. The case of $p = 0$ (or $p = 1$) has already been worked out in Ref. 12. However, the case of generic p was not worked out.

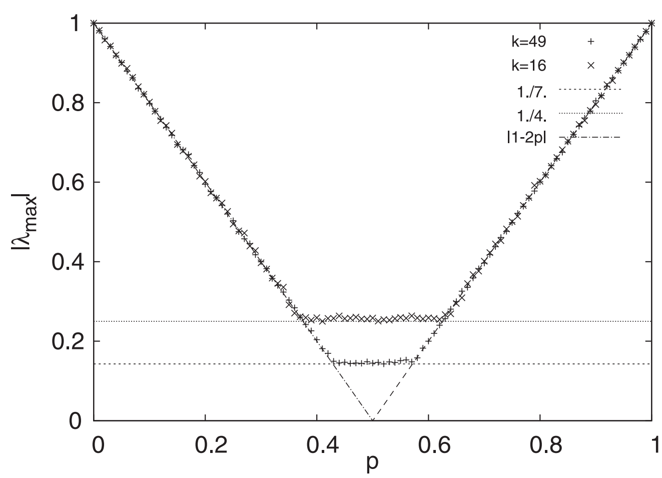

FIG. 4. Magnitude of largest eigenvalue of adjacency matrix is plotted as a function of p for $k = 16$ and $k = 49$. Lines at $1/\sqrt{k}$ are also plotted as reference. It is clear that eigenvalue is $\max((1 - 2p), 1/\sqrt{k})$.

First, we study the most random case $p_1 = p_2 = 1/2$ (and thus $p = 1/2$). This matrix can be viewed as a matrix with 0 entries having probability $1 - 2r$ and entries $+1/k$ or $-1/k$ having probability $r = k/(2n)$ each. This distribution has a zero mean and the variance is $\sigma^2 = 1/kn$. Thus $\sigma\sqrt{n} = 1/\sqrt{k}$. This distribution clearly has a bounded $2+\delta$ moment for $\delta > 0$. The theorem by Tao and Vu suggests that eigenvalues of $A\sqrt{k}$ will fall within an unit disc in the complex plane. Thus, the largest absolute value that eigenvalues of A can take is $1/\sqrt{k}$.[13,14]

For $p \neq 1/2$, the entries do not have mean zero any longer and most random matrix results do not apply. Random matrices with nonzero mean have been studied as far back as 1964.[15] Later, it was understood that such matrices could be seen as rank one perturbation of random matrix with zero mean. In our case, we can write a given matrix as $A = B + P$ where $B_{i,j} = A_{i,j} - \mu/n$ and $P_{i,j} = \mu/n$ where μ/n where $\mu = (1-2p)$ is the expected sum of row elements and μ/n is the mean. For $A_{i,j} = 1/k, 0$ and $-1/k$, $B_{i,j} = 1/k - \mu/n, -1/n$ and $-1/k - \mu/n$ respectively. Thus matrix B is an iid random matrix with zero mean. The matrix P can be written as $P = \mu|e><e|$ where $|e> = n^{-1/2}1_n$.

Recent studies have shown that random iid matrices with nonzero mean have an outlier when the mean is large enough.[16] Theorem 1.4 and 1.7 in Ref. 16 explain our observations very well if we assume that the theorem works for sparse matrices as well and recast the matrices appropriately. Sparse matrices differ from standard iid matrices since the mean and variance of entries are not independent of n. Consider the matrix B defined above. For B, the variance of the entries is $q(1-p)(1/k - \mu/n)^2 + qp(-1/k - \mu/n)^2 + (1-q)(-\mu/n)^2$ where $q = k/n$. For $k \ll n$, $\sigma^2 \sim 1/(kn)$. Hence matrix $B/\sigma = \sqrt{nk}B$ has entries with unit variance and zero mean. This is X_n in Tao's notation. Now Theorem 1.4 of Tao implies that $X_n/\sqrt{n} = \sqrt{k}B$ has eigenvalues within the unit circle. We know that $\sqrt{k}A = \sqrt{k}B + \sqrt{k}P$. The perturbation matrix $C_n = \sqrt{k}P$ has rank-one: One eigenvalue is $(1-2p)\sqrt{k}$ and the remaining ones are zero. Theorem 1.7 of Tao[16] gives the eigenvalue distribution for matrix $X_n/\sqrt{n} + C_n = \sqrt{k}B + \sqrt{k}P = \sqrt{k}A$. $n-1$ eigenvalues of $\sqrt{k}A$ lie within the unit circle while one is given by the eigenvalue of C_n, i.e., $(1-2p)\sqrt{k}$ provided it is outside the unit circle. It follows that $n-1$ eigenvalues of A are within a circle of radius $1/\sqrt{k}$ and the one outside is given by $(1-2p)$ if $|1-2p| > 1/\sqrt{k}$. Otherwise, all the eigenvalues of A lie within a circle of radius $1/\sqrt{k}$.[16] This latter result explains the flattening of the bottom part of the inverted pyramid in Fig. 1.

Instead of treating the adjacency matrix as a perturbation from a zero mean iid matrix, we can take a different viewpoint and try out first order perturbation theory starting from the adjacency matrix for $p = 0$. Let us write $A = B' + C'$ where B' is the adjacency matrix for $p = 0$, i.e., all nonzero entries are positive. We know that B''s largest eigenvalue is 1 with right eigenvector $e_r = n^{-1/2}[1, 1, \ldots 1] = |e>$. What is the left eigenvector of B? The matrix B'^T can be viewed as a matrix with nonzero entries $1/k$ with probability k/n and 0 entries with probability $1 - k/n$. The right eigenvector of such a matrix is close to vector $|e>$.[17] Thus we can assume that both the left and right largest eigenvectors of B' are given by $|e>$ The nonzero entries of C' are $-2/k$ with probability pk/n and other entries are zero. Hence, first order perturbation in the largest eigenvalue of B' obtained by adding C' is given by $\langle e|C'|e\rangle/\langle e|e\rangle = -2p$. Thus near $p = 0$, the largest eigenvalue is $(1-2p)$.

IV. COUPLED OSCILLATORS

Next we try to understand how this work translates to coupled oscillators. We couple oscillators in exactly the same manner as earlier

$$\dot{\boldsymbol{x}}_i(t+1) = \sum_{j=1}^{n} A_{i,j} F(\boldsymbol{x}_j(t)), \tag{5}$$

where $\boldsymbol{x}$ is a M dimensional vector, the function $F(\boldsymbol{x})$ is such that $F(0) = 0$. This system has an amplitude death state as a possible state. We investigate the stability of such a system. When we linearize around the amplitude death state, the resultant Jacobian is a direct product of connectivity matrix A and the Jacobian of a single oscillator. One may guess that the eigenvalues of such a Jacobian are a direct product of eigenvalues of the connectivity matrix A and eigenvalues of the Jacobian for a single system. If a single linearized system has (say) real eigenvalues r_1, r_2, and r_3, and if coupling matrix A is of type studied in Sec. III with (say) $p = 0$, we expect the eigenvalues of the linearized system to be r_1, r_2, r_3 and $3n - 3$ eigenvalues (from the connectivity matrix) which are located within circles in the complex plane with radii $\frac{|r_1|}{\sqrt{k}}, \frac{|r_2|}{\sqrt{k}}, \frac{|r_3|}{\sqrt{k}}$. This expectation is indeed fulfilled. For a standard Lorenz system with $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$, eigenvalues of the linearized (single) system are -22.82, 11.82 and $-8/3$. We observe that for a coupled system with $k = 9$, the three eigenvalues of the individual system are retained and other eigenvalues are located within circles in the complex plane with radii 22.82/3, 11.82/3, and 8/9. For $p = 0.5$, we only see eigenvalues inside these three circles centered at origin in complex plane. Real part of these eigenvalues can be positive or negative. The eigenvalue spectrum for $n = 500$ is shown in Fig. 5. Since the real parts of all eigenvalues need to be negative for stability, it is clear that amplitude death state cannot be stabilized with such connectivity.

We try another connectivity. Let us consider coupled Lorenz systems with dynamics given by

$$\begin{aligned}
\dot{x}_i &= \sigma(y_i - x_i) + \sum_{j=1}^{N} \epsilon(A_{i,j}x_j - |A_{i,j}|x_i),\\
\dot{y}_i &= \rho x_i - y_i + \sum_{j=1}^{N} \epsilon(A_{i,j}y_j - |A_{i,j}|y_i),\\
\dot{z}_i &= x_i y_i - \beta z_i.
\end{aligned} \tag{6}$$

For $\epsilon = 0$, the dynamics is that of a single uncoupled oscillator. We choose $\epsilon = 2$, $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The coupling matrix A is same as discussed in Sec. III with $k = 9$.

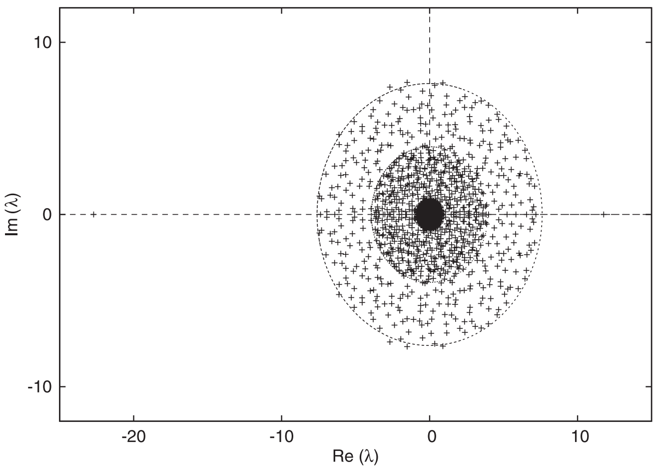

FIG. 5. Eigenvalues of the Jacobian of amplitude death state of coupled Lorenz oscillators with random connectivity is plotted for $p = 0$, $n = 500$ and $k = 9$. We have also drawn circles centred at origin with radii $|r_1|/3$, $|r_2|/3$, and $|r_3|/3$ for reference. (Eigenvalues of single Lorenz oscillator are given by $r_1 = -22.82$, $r_2 = 11.82$ and $r_3 = -8/3$.) Three eigenvalues of original oscillator are retained and $3n - 3$ eigenvalues are in these circles.

Eigenvalues of this system are not obvious. However, some features of the eigenvalues of matrix A are carried over. We have seen that the largest eigenvalue for amplitude death state is non-degenerate for $p = 0$. This eigenvalue belongs to the eigenvector $[1, 1, \ldots 1]$. Hence, we expect the dynamics to be close to synchronized state for small values of p and desynchronized dynamics for larger values of p. This expectation is indeed fulfilled. For $p = 0$, we observe synchronized state. For small p, the system is not exactly synchronized. However, all the systems are clustered in a small part of phase space. In fact, we find sub-clusters within this cluster at a finer scale. On further increasing p, we find synchronized as well as anti-synchronized clusters. For $p = 1$, we find synchronized as well as anti-synchronized clusters in equal proportion. Unlike coupled maps, the cases of $p = 0$ and $p = 1$ are no longer equivalent for coupled oscillators. Single Lorenz system has a large positive eigenvalue at 11.82 and from eigenvalue analysis, amplitude death state is stabilized for large values of ϵ and $p > 0$ (say $\epsilon = 60$ and $p = 1$.) Thus, we observe a synchronization-desynchronization transition induced by negative coupling or even amplitude death for large perturbation.

We would like to note another difference between the two coupling schemes studied above. Given arbitrary differential equations, it is not guaranteed that the system has an attractor. Lorenz system has an attractor, i.e., the trajectories are confined to a part of phase space and do not go to infinity. If we have a collection of N uncoupled Lorenz systems, it will have attractor as well. We can hope that, for very small couplings, the coupled system will continue to have an attractor. In our first example, we cannot have any limiting case in which system is reduced to N uncoupled Lorenz equations for the given connectivity matrix. However, in the second example, for $\epsilon = 0$ the system reduces to N uncoupled Lorenz oscillators and consequently the system has an attractor. Hence, we find that the trajectories do not escape to infinity in this system which is a continuous deformation of the system of N uncoupled Lorenz oscillators.

Our results are somewhat different from those in Resmi *et al.*[2] The main difference is that they have an extra dynamical variable which can change the stability of original system. Thus, the phase space of two coupled Lorenz oscillators is not six dimensional but seven dimensional owing to the extra control variable in the above Ref. 2. Dynamics of this control variable is very similar to adaptive control of chaos which has been used for a single oscillator.[18] We have concentrated on possible dynamics of the system solely in presence of different types of couplings and without the extra control variable.

V. CONCLUSIONS

We have studied the stability of the oscillator/amplitude death state of a coupled map lattice in which we have two sublattices/groups. If all intra-group connections are of the same sign and if inter-group connections are also consistently of the same sign, the oscillators do not get frozen. We can observe chaotic synchronization and chaotic antisynchronization in this case. However, when we do not have this consistency we observe oscillator death which is most pronounced when the couplings have random signs. We have used random matrix theory arguments to show that in many cases there is an oscillator death in the system. We have focused primarily on a random network in this paper. However, we expect similar results to hold even for other networks. In particular, introduction of a few negative elements in a positive matrix will reduce spectral radius of a matrix in general. We have also briefly commented on a possible synchronization-desynchronization transition in coupled oscillators in the presence of negative couplings.

We would like to note that for competing interactions, Antal *et al.*[8] have given a concept of balanced network. A balanced network is one in which all its constituent triangles are balanced. Interestingly, such a balanced network falls within the class of networks we have studied in this work. As Antal *et al.* note, "A fundamental result from these studies is that balanced societies are remarkably simple: either all individuals are mutual friends (we call such a state 'paradise'), or the network segregates into two antagonistic cliques where individuals within the same clique are mutual friends and individuals from distinct cliques are enemies (we call such a state 'bipolar').[8]" We find that such balanced "paradise" or "bipolar" connectivity induces a synchronized or antisynchronized state while imbalanced/inconsistent connectivity leads to oscillator death. In the context of social networks, if an individual has a choice of not taking any side in a conflict, all individuals are likely to have no opinion in a frustrated network. On the other hand, all of them will hold the same opinion in a "paradise" network and will split into two groups of diverging opinions in a "bipolar" network.

A physical system that fits into our framework is the metamagnet introduced earlier. Instead of an Ising system, the Blume-Capel model could be simulated on such connectivities.[19] In Blume-Capel model, spin can take $+1$, -1 and 0 values. It is clear that ferromagnetic state will be a low energy state if all connections are positive and antiferromagnetic state with opposing magnetization will be a low energy state if we have two groups with ferromagnetic intra-group and antiferromagnetic inter-group interactions. However, it would be of interest if we can reach an entirely nonmagnetic (spin 0) state on a frustrated network.

ACKNOWLEDGMENTS

P.M.G. was supported by DST grant. G.R. was supported by JC Bose National Fellowship, DST Centre for Mathematical Biology, DST IRHPA Centre for Neuroscience and UGC Centre for Advanced Studies. G.R. is also a Honorary Professor at the JNCASR, Bangalore, India.

[1]G. Saxena, A. Prasad, and R. Ramaswamy, Phys. Rep. **521**, 205 (2012).
[2]V. Resmi, G. Ambika, and R. E. Amritkar, Phys. Rev. E **84**, 046212 (2011).
[3]A. V. Rangan and D. Cai, Phys. Rev. Lett. **96**, 178101 (2006).
[4]X. Chen and J. E. Cohen, J. Theor. Biol.**212**, 223 (2001).
[5]See, e.g., T. W. Carr, M. L. Taylor, and I. B. Schwartz, Physica D **213**, 152 (2006).
[6]M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[7]P. Bruisma and G. Aeppli, Phys. Rev. B **29**, 2644 (1984).
[8]T. Antal, P. L. Krapivsky, and S. Redner, Phys. Rev. E **72**, 036121 (2005).
[9]J. Miller and D. Huse, Phys. Rev. E **48**, 2528 (1993).
[10]For a model with both positive and negative connections in spin systems, see D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[11]P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[12]P. M. Gade, Phys. Rev. E **54**, 64, (1996).
[13]P. M. Gade and C.-K. Hu, Phys. Rev. E **62**, 6409 (2000) and references therein.
[14]T. Tao and V. Vu, Commun. Contemp. Math. **10.02**, 261 (2008); e-print arXiv:0708.2895 [math.PR].
[15]D. W. Lang, Phys. Rev. **135**, B1082, (1964).
[16]T. Tao, Probab. Theory and Related Fields **155**, 231 (2013); e-print arXiv:1012.4818v5 [math.PR.].
[17]P. Mitra, Electron. J. Comb. **16**, R131 (2009).
[18]K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. E **70**, 026215 (2004).
[19]See e. g., D. M. Saul, M. Wortis, and D. Stauffer, Phys. Rev. B **9**, 4964 (1974).