LONG TERM STABILITY STUDIES OF PARTICLE STORAGE RINGS USING POLYNOMIAL MAPS

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Abstract Long-term stability studies of particle storage rings can not be carried out using conventional numerical integration algorithms. We require symplectic integration algorithms which are both fast and accurate. In this paper, we study a symplectic integration method wherein the symplectic map representing the Hamiltonian system is refactorized using polynomial symplectic maps. This method is used to perform long term integration on a particle storage ring.

Keywords: Symplectic integration, polynomial maps, stability

1. Introduction

Standard numerical integration algorithms can not be used to study long term stability of particle storage rings since they are not symplectic [1]. This violation of the symplectic condition can lead to spurious chaotic or dissipative behavior. Numerical integration algorithms which satisfy the symplectic condition are called symplectic integration algorithms [1]. Over the last few years, there has been considerable efforts devoted to developing such symplectic integration algorithms [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In our approach, we use the the symplectic map [22, 23] representing the particle storage ring. For complicated storage rings like the Large Hadron Collider which has thousands of elements, using individual Hamiltonians for each element can drastically slow down the integration process. On the other hand, the map based approach is very fast in such cases [24, 25].

We investigate a new symplectic integration method where the symplectic map is refactorized using "polynomial maps" [27, 28]. This method has the advantage of not introducing spurious poles and branch points. Further, since it is map-based, it is also very fast.

2. Preliminaries

We restrict ourselves to three degrees of freedom nonlinear Hamiltonian system. We start by defining certain mathematical objects. Let us denote the collection of six phase-space variables q_i, p_i $(i = 1, 2, 3)$ by the symbol z:

$$z = (q_1, p_1, q_2, p_2, q_3, p_3). \tag{1}$$

The Lie operator corresponding to a phase-space function $f(z)$ is denoted by $:f(z):$. It is defined by its action on a phase-space function $g(z)$ as shown below

$$:f(z):g(z) = [f(z), g(z)]. \tag{2}$$

Here $[f(z), g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$. Next, we define the exponential of a Lie operator. It is called a Lie transformation and is given as follows:

$$e^{:f(z):} = \sum_{n=0}^{\infty} \frac{:f(z):^n}{n!}. \tag{3}$$

Powers of $:f(z):$ that appear in the above equation are defined recursively by the relation

$$:f(z):^n g(z) = :f(z):^{n-1}[f(z), g(z)], \tag{4}$$

with

$$:f(z):^0 g(z) = g(z). \tag{5}$$

For further details regarding Lie operators and Lie transformations, see Ref [22].

The time evolution of the Hamiltonian system over one period can be represented by a symplectic map $\mathcal{M}$[22]. Symplectic maps are maps whose Jacobian matrices $M(z)$ satisfy the following symplectic condition

$$\widetilde{M(z)} J M(z) = J \tag{6}$$

where $\tilde{M}$ is the transpose of M and J is an antisymmetric matrix defined as follows:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7}$$

Matrices M satisfying Eq. (6) are called symplectic matrices and the corresponding maps $\mathcal{M}$ symplectic maps. It can be shown[22] that the set of all $\mathcal{M}$'s forms an infinite dimensional Lie group of symplectic maps. On the other hand, the set of all real 6×6 symplectic matrices forms the finite dimensional real symplectic group Sp(6,R).

Using the Dragt-Finn factorization theorem[22, 29], the symplectic map $\mathcal{M}$ can be factorized as shown below:

$$\mathcal{M} = \hat{M} e^{:f_3:} e^{:f_4:} \ldots e^{:f_n:} \ldots \tag{8}$$

Here $\hat{M}$ gives the linear part of the map and hence has an equivalent representation in terms of the Jacobian matrix $M(0)$ of the map $\mathcal{M}$ at the origin[22]:

$$\hat{M} z_i = M_{ij} z_j = (Mz)_i. \tag{9}$$

Thus, $\hat{M}$ is said to be the Lie transformation corresponding to the 6×6 matrix M belonging to Sp(6,R). The infinite product of Lie transformations $\exp(:f_n:)$ $(n = 3, 4, \ldots)$ in Eq. (8) represents the nonlinear part of $\mathcal{M}$. Here $f_n(z)$ denotes a homogeneous polynomial (in z) of degree n uniquely determined by the factorization theorem.

As an application, let us consider a charged particle particle storage ring which typically comprises thousands of elements (drifts, quadrupoles, sextupoles etc.) Using the above procedure, one can represent each element in the storage ring by a symplectic map. By concatenating [22] these maps together using group-theoretical methods [30], we obtain the so-called 'one-turn' map representing the entire storage ring. The one-turn map gives the final state $z^{(1)}$ of a particle after one turn around the ring as a function of its initial state $z^{(0)}$:

$$z^{(1)} = \mathcal{M} z^{(0)}. \tag{10}$$

To obtain the state of a particle after n turns, one has to merely iterate the above mapping N times i.e.

$$z^{(n)} = \mathcal{M}^n z^{(0)}. \tag{11}$$

Since $\mathcal{M}$ is explicitly symplectic, this gives a symplectic integration algorithm. Further, since the entire ring can be represented by a single (or at most a few) symplectic map(s), numerical integration of particle trajectories using symplectic maps is very fast.

To obtain a practical symplectic integration algorithm, we follow the perturbative approach and truncate $\mathcal{M}$ after a finite number of Lie transformations:

$$\mathcal{M} \approx \hat{M} e^{:f_3:} e^{:f_4:} \ldots e^{:f_P:}. \tag{12}$$

The symplectic map is said to be truncated at order P. This map is still symplectic. However, each exponential $e^{:f_n:}$ in $\mathcal{M}$ still contains an infinite number of terms in its Taylor series expansion. We get around the above problem by refactorizing $\mathcal{M}$ in terms of simpler symplectic maps which can be evaluated exactly without truncation. We use 'polynomial maps' which give rise to polynomials when acting on the phase space variables. This avoids the problem of spurious poles and branch points present in generating function methods [26], solvable map [12, 21] and monomial map [18] refactorizations.

3. Symplectic Polynomial Maps

We start by describing the difference between monomial maps and polynomial maps with respect to presence of poles and branch points. This difference can be illustrated using the following examples. Consider the monomial symplectic map $\exp(:q_1^2p_1:)$. Its action on q_1, p_1 in a two dimensional phase space is given as follows:

$$q_1' = \exp(:q_1^2p_1:)q_1 = \frac{q_1}{1+q_1}; \quad p_1' = \exp(:q_1^2p_1:)p_1 = p_1(1+q_1)^2. \quad (13)$$

This map has a pole at $q_1 = -1$.

On the other hand, consider the symplectic map $\exp(:a_1q_1^3 + a_2p_1:)$ where a_1, a_2 are real constants. We determine its action on phase space variables as follows. Note that the symplectic map is of the form $\exp(:h(z):)$ where $h(z)$ is a function which depends only on the phase space variables z and is independent of time t. If we take $h(z)$ to be the Hamiltonian function, then solving the Hamilton's equations of motion for this Hamiltonian from time $t = t^i$ to time $t = t^f$ is equivalent to the following symplectic map action [22]:

$$z(t = t^f) = \exp[-(t^f - t^i) : h(z) :]z(t = t^i). \quad (14)$$

Equivalently, obtaining the action of the symplectic map $\exp[-(t^f - t^i) : h(z) :]$ on the phase space variables is the same as solving the Hamilton's equations of motion with $h(z)$ as the Hamiltonian from time t^i to t^f. Setting $t^i = 0$ and $t^f = -1$ we have the following equivalence: Obtaining the action of the symplectic map $\exp(: h(z :)$ on phase space variables is equivalent to solving the Hamilton's equations of motion using $h(z)$ as the Hamiltonian from time $t = 0$ to time $t = -1$. In this case, $z(0)$ will correspond to the initial values of the phase space variables and $z(-1)$ to the final values obtained after the action of the map $\exp(: h(z :)$. Returning to our symplectic map, we obtain its action using the above procedure. Thus we get

$$q_1^{fin} = q_1^{in} - a_2, \quad p_1^{fin} = p_1^{in} + a_1a_2^2 - 3a_1a_2q_1^{in} + 3a_1(q_1^{in})^2. \quad (15)$$

We note that the final values of the phase space variables are polynomial functions of the initial variables and therefore involve no poles or branch points. This is an example of a polynomial map.

We now turn to the question of which symplectic maps have polynomial action. It can be shown [31] that the following results are true

1 All polynomials of the form $h(z)$ where both a phase space variable and its canonically conjugate variable [32] do not occur simultaneously give rise to symplectic polynomial maps via exp(: $h(z)$:). We will call such $h(z)$'s as polynomials of the first type.

2 If a canonically conjugate pair q_i, p_i is present in the polynomial $h(z)$ and it appears either in the form $[a(\bar{z})q_i+g(p_i,\bar{z})]^m$ or $[a(\bar{z})p_i+g(q_i,\bar{z})]^m$ (where $m = 1, 2 \ldots$, $\bar{z} = \{q_j, p_k\}$ with $j \neq k \neq i$ and a, g are polynomials in the indicated variables), then this polynomial $h(z)$ again gives rise to a symplectic polynomial map via exp(: $h(z)$:). If a product/sum of such factors appears in $h(z)$, each term in the product/sum is a function of different canonically conjugate pairs. We will call $h(z)$'s of the form described above as polynomials of the second type.

4. Symplectic Integration using Polynomial Maps

In this section, we return to the problem of symplectic integration. We restrict ourselves to symplectic maps in a six dimensional phase space truncated at order 4. The results obtained below can be generalized to both higher orders and higher dimensions using symbolic manipulation programs. The Dragt-Finn factorization of the symplectic map is given by:

$$\mathcal{M} = \hat{M} e^{:f_3:} e^{:f_4:}, \tag{16}$$

where

$$\begin{aligned} f_3 &= a_{28}q_1^3 + a_{29}q_1^2 p_1 + \cdots + a_{83}p_3^3, \\ f_4 &= a_{84}q_1^4 + a_{85}q_1^3 p_1 + \cdots + a_{209}p_3^4. \end{aligned} \tag{17}$$

Here the coefficients $a_{28}, \ldots, a_{209}$ can be explicitly computed given a Hamiltonian system[22] and are therefore known to us. The numbering of these monomial coefficients follows the standard Giorgilli scheme [33]. The above map captures the leading order nonlinearities of the system. Since the action of the linear part $\hat{M}$ on phase space variables is well known [cf. Eq. (9)] and is already a polynomial action, we only refactorize the nonlinear part of the map using N polynomial maps [27]. This is done as follows:

$$\mathcal{M} \approx \mathcal{P} = \hat{M} e^{:h_1:} e^{:h_2:} \cdots e^{:h_N:}, \tag{18}$$

where $e^{:h_i:}$'s are symplectic polynomial maps and the numeral appearing in the subscript indexes the polynomial maps. The polynomial maps are determined by requiring that $\mathcal{P}$ agree with $\mathcal{M}$ up to order 4. That is, when the N polynomial maps are combined, the resulting symplectic map should have all the monomials present in f_3 and f_4 with the correct coefficients up to order 4.

Using the above procedure, it turns out that we require 23 polynomial maps for refactorization:

$$\mathcal{M} \approx \mathcal{P} = \hat{M} e^{:h_1:} e^{:h_2:} \cdots e^{:h_{23}:}, \tag{19}$$

The h_i's are given as follows:

$$\begin{aligned}
h_1 &= q_1^3\, b_{28} + q_1^2\, q_2\, b_{30} + q_1^2\, q_3\, b_{32} + q_1\, q_2^2\, b_{39} + q_1\, q_2\, q_3\, b_{41} + q_1\, q_3^2\, b_{46} + \\
&\quad q_2^3\, b_{64} + q_2^2\, q_3\, b_{66} + q_2\, q_3^2\, b_{71} + q_3^3\, b_{80} + q_1^4\, b_{84} + q_1^3\, q_2\, b_{86} + q_1^3\, q_3\, b_{88} + \\
&\quad q_1^2\, q_2^2\, b_{95} + q_1^2\, q_2\, q_3\, b_{97} + q_1^2\, q_3^2\, b_{102} + q_1\, q_2^3\, b_{120} + q_1\, q_2^2\, q_3\, b_{122} + \\
&\quad q_1\, q_2\, q_3^2\, b_{127} + q_1\, q_3^3\, b_{136} + q_2^4\, b_{175} + q_2^3\, q_3\, b_{177} + q_2^2\, q_3^2\, b_{182} + \\
&\quad q_2\, q_3^3\, b_{191} + q_3^4\, b_{205}, \\
h_2 &= [(b_{29} + b_{34}) + q_2\ (b_{91} + b_{106}) + p_2\ (b_{92} + b_{107}) + q_3\ (b_{93} + b_{108}) + \\
&\quad p_3\ (b_{94} + b_{109})]\,(p_1 + q_1)^3, \\
h_3 &= [(-b_{29} + b_{34}) + q_2\ (-b_{91} + b_{106}) + p_2\ (-b_{92} + b_{107}) + \\
&\quad q_3\ (-b_{93} + b_{108}) + p_3\ (-b_{94} + b_{109})]\,(-p_1 + q_1)^3, \\
h_4 &= [(b_{65} + b_{68}) + q_1\ (b_{121} + b_{124}) + p_1\ (b_{156} + b_{159}) + \\
&\quad q_3\ (b_{180} + b_{186}) + p_3\ (b_{181} + b_{187})]\,(p_2 + q_2)^3, \\
h_5 &= [(-b_{65} + b_{68}) + q_1\ (-b_{121} + b_{124}) + p_1\ (-b_{156} + b_{159}) + \\
&\quad q_3\ (-b_{180} + b_{186}) + p_3\ (-b_{181} + b_{187})]\,(-p_2 + q_2)^3, \\
h_6 &= [(b_{81} + b_{82}) + q_1\ (b_{137} + b_{138}) + p_1\ (b_{172} + b_{173}) + q_2\ (b_{192} + b_{193}) + \\
&\quad p_2\ (b_{202} + b_{203})]\,(p_3 + q_3)^3, \\
h_7 &= [(-b_{81} + b_{82}) + q_1\ (-b_{137} + b_{138}) + p_1\ (-b_{172} + b_{173}) + \\
&\quad q_2\ (-b_{192} + b_{193}) + p_2\ (-b_{202} + b_{203})]\,(-p_3 + q_3)^3, \\
h_8 &= (p_1 + q_1)^2 \left(q_2\, b_{35} + q_3\, b_{37} + q_2^2\, b_{110} + q_2\, q_3\, b_{112} + p_3\, q_2\, b_{113} + q_3^2\, b_{117}\right), \\
h_9 &= (p_1 + q_1)^2 \left(p_2\, b_{36} + p_3\, b_{38} + p_2^2\, b_{114} + p_2\, q_3\, b_{115} + p_2\, p_3\, b_{116} + p_3^2\, b_{119}\right), \\
h_{10} &= (p_2 + q_2)^2 \left(q_1\, b_{40} + q_3\, b_{69} + q_1^2\, b_{96} + q_1\, q_3\, b_{125} + p_3\, q_1\, b_{126} + q_3^2\, b_{188}\right), \\
h_{11} &= (p_2 + q_2)^2 \left(p_1\, b_{55} + p_3\, b_{70} + p_1^2\, b_{146} + p_1\, q_3\, b_{160} + p_1\, p_3\, b_{161} + p_3^2\, b_{190}\right),
\end{aligned}$$

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\begin{aligned}
h_{12} &= (p_3+q_3)^2 \left(q_1\, b_{47} + q_2\, b_{72} + q_1^2\, b_{103} + q_1\, q_2\, b_{128} + p_2\, q_1\, b_{134} + q_2^2\, b_{183}\right), \\
h_{13} &= (p_3+q_3)^2 \left(p_1\, b_{62} + p_2\, b_{78} + p_1^2\, b_{153} + p_1\, q_2\, b_{163} + p_1\, p_2\, b_{169} + p_2^2\, b_{199}\right), \\
h_{14} &= p_2\, q_1^2\, b_{31} + p_3\, q_1^2\, b_{33} + p_2^2\, q_1\, b_{43} + p_2\, p_3\, q_1\, b_{45} + p_3^2\, q_1\, b_{48} + p_2\, q_1^3\, b_{87} + \\
&\quad p_3\, q_1^3\, b_{89} + p_2^2\, q_1^2\, b_{99} + p_2\, p_3\, q_1^2\, b_{101} + p_3^2\, q_1^2\, b_{104} + p_2^3\, q_1\, b_{130} + \\
&\quad p_2^2\, p_3\, q_1\, b_{132} + p_2\, p_3^2\, q_1\, b_{135} + p_3^3\, q_1\, b_{139}, \\
h_{15} &= p_1^2\, q_2\, b_{50} + p_1^2\, q_3\, b_{52} + p_1\, q_2^2\, b_{54} + p_1\, q_2\, q_3\, b_{56} + p_1\, q_3^2\, b_{61} + p_1^3\, q_2\, b_{141} + \\
&\quad p_1^3\, q_3\, b_{143} + p_1^2\, q_2^2\, b_{145} + p_1^2\, q_2\, q_3\, b_{147} + p_1^2\, q_3^2\, b_{152} + p_1\, q_2^3\, b_{155} + \\
&\quad p_1\, q_2^2\, q_3\, b_{157} + p_1\, q_2\, q_3^2\, b_{162} + p_1\, q_3^3\, b_{171}, \\
h_{16} &= p_1\, p_3\, q_2\, b_{57} + p_3\, q_2^2\, b_{67} + p_3^2\, q_2\, b_{73} + p_1^2\, p_3\, q_2\, b_{148} + p_1\, p_3\, q_2^2\, b_{158} + \\
&\quad p_1\, p_3^2\, q_2\, b_{164} + p_3\, q_2^3\, b_{178} + p_3^2\, q_2^2\, b_{184} + p_3^3\, q_2\, b_{194}, \\
h_{17} &= p_2\, q_1\, q_3\, b_{44} + p_2^2\, q_3\, b_{75} + p_2\, q_3^2\, b_{77} + p_2\, q_1^2\, q_3\, b_{100} + p_2^2\, q_1\, q_3\, b_{131} + \\
&\quad p_2\, q_1\, q_3^2\, b_{133} + p_2^3\, q_3\, b_{196} + p_2^2\, q_3^2\, b_{198} + p_2\, q_3^3\, b_{201}, \\
h_{18} &= p_1\, p_2\, q_3\, b_{59} + p_1^2\, p_2\, q_3\, b_{150} + p_1\, p_2^2\, q_3\, b_{166} + p_1\, p_2\, q_3^2\, b_{168}, \\
h_{19} &= p_3\, q_1\, q_2\, b_{42} + p_3\, q_1^2\, q_2\, b_{98} + p_3\, q_1\, q_2^2\, b_{123} + p_3^2\, q_1\, q_2\, b_{129}, \\
h_{20} &= p_1^3\, b_{49} + p_1^2\, p_2\, b_{51} + p_1^2\, p_3\, b_{53} + p_1\, p_2^2\, b_{58} + p_1\, p_2\, p_3\, b_{60} + p_1\, p_3^2\, b_{63} + \\
&\quad p_2^3\, b_{74} + p_2^2\, p_3\, b_{76} + p_2\, p_3^2\, b_{79} + p_3^3\, b_{83} + p_1^4\, b_{140} + p_1^3\, p_2\, b_{142} + \\
&\quad p_1^3\, p_3\, b_{144} + p_1^2\, p_2^2\, b_{149} + p_1^2\, p_2\, p_3\, b_{151} + p_1^2\, p_3^2\, b_{154} + p_1\, p_2^3\, b_{165} + \\
&\quad p_1\, p_2^2\, p_3\, b_{167} + p_1\, p_2\, p_3^2\, b_{170} + p_1\, p_3^3\, b_{174} + p_2^4\, b_{195} + p_2^3\, p_3\, b_{197} + \\
&\quad p_2^2\, p_3^2\, b_{200} + p_2\, p_3^3\, b_{204} + p_3^4\, b_{209}, \\
h_{21} &= \left(p_1 + q_1 + p_1^2\, b_{105}\right)^3 + \left(p_2 + q_2 + p_2^2\, b_{185}\right)^3 + \left(p_3 + q_3 + p_3^2\, b_{208}\right)^3, \\
h_{22} &= \left(-p_1 - q_1 + q_1^2\, b_{85}\right)^3 + \left(-p_2 - q_2 + q_2^2\, b_{176}\right)^3 + \\
&\quad \left(-p_3 - q_3 + q_3^2\, b_{206}\right)^3, \\
h_{23} &= (p_1+q_1)^4\, b_{90} + (p_1+q_1)^2\,(p_2+q_2)^2\, b_{111} + (p_1+q_1)^2\,(p_3+q_3)^2\, b_{118} \\
&\quad + (p_2+q_2)^4\, b_{179} + (p_2+q_2)^2\,(p_3+q_3)^2\, b_{189} + (p_3+q_3)^4\, b_{207}.
\end{aligned}
$$

Here b_i's are at present unknown coefficients. As mentioned above, by forcing the refactorized form $\mathcal{P}$ to equal the original map $\mathcal{M}$ up to order 4 and using the CBH theorem[30], we can easily compute these unknown coefficients in terms of the known a_i's. These expressions are available from the author as part of a FORTRAN program implementing the above algorithm.

The explicit actions of the polynomial maps on phase space variables can be obtained. This completely determines the refactorized map $\mathcal{P}$. Each $\exp(:h_i:)$ is a polynomial map which can be evaluated exactly and is explicitly symplectic. Thus by using $\mathcal{P}$ instead of $\mathcal{M}$ in Eq. (11), we obtain an explicitly symplectic integration algorithm. Further, it is fast to evaluate and does not introduce spurious poles and branch points. The above factorization is not unique. However, the principles outlined earlier impose restrictions on the possible forms and this eases considerably the task of refactorization. Moreover, we require the coefficients b_i to be polynomials in the known coefficients a_i. Otherwise this can lead to divergences when a_i's take on certain special values. Finally, we minimize the number of polynomial maps in the refactorized form. Our studies show that different polynomial map refactorizations obeying the above restrictions do not lead to any significant differences in their behavior.

5. Applications

We now consider two applications of the above method. The first example is to find the region of stability of the following simple symplectic map:

$$\mathcal{M} = \hat{M} \exp[:(q_1 + p_1)^3 :], \tag{20}$$

where

$$\hat{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{21}$$

and $\theta = \frac{\pi}{3}$. We chose this example since the exact action of the above map is known and hence the exact region of stability can also be determined. We found excellent agreement between results obtained using polynomial maps and the exact results.

We have also applied the method to a large particle storage ring for storing charged particles. This storage ring consists of 5109 individual elements (where these elements could be drifts, bending magnets, quadrupoles or sextupoles). If one tries to numerically integrate the trajectory of a charged particle through this ring using a conventional integration algorithm, one has to go through the ring element by element where each element is described by its own Hamiltonian. This is cumbersome and slow and further, does not respect the Hamiltonian nature of the system. On the other hand, a map based approach where one represents the entire storage ring in terms of a single map is much faster [24, 25]. When this is combined with our polynomial map refactorization, one obtains a symplectic integration algorithm which is both fast and accurate and is ideally suited for such complex real life systems. The $q_1 - p_1$ phase plot for one million turns around the ring using our polynomial map method is given in Figure 1. In this case, q_1 and p_1 represent the deviations from the closed orbit coordinate and momentum

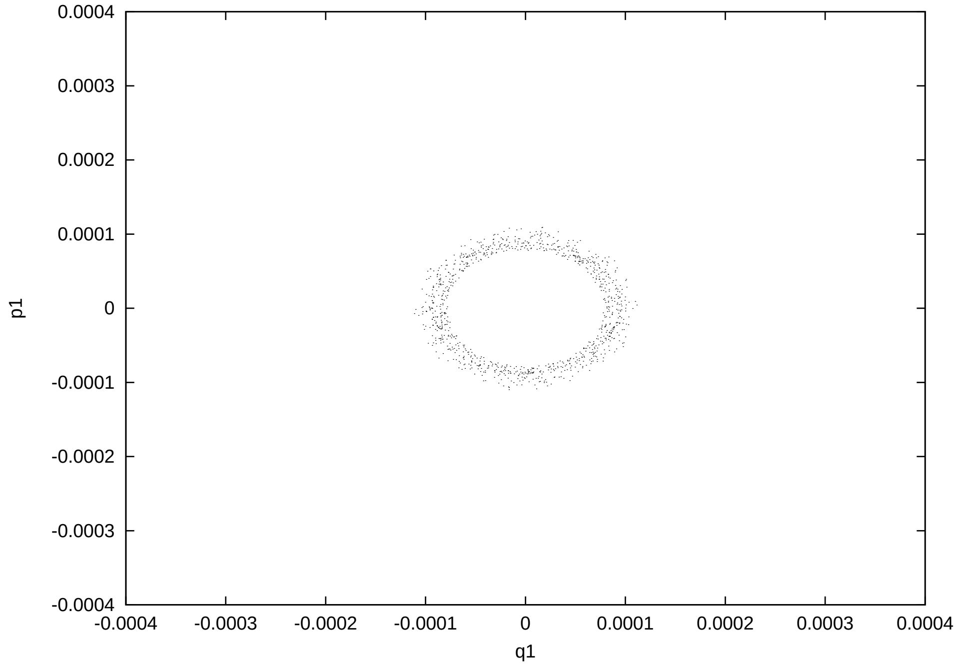

Figure 1. This figure shows the $q_1 - p_1$ phase space plot for one million turns around a storage ring using the polynomial map method (only every 1000th point is plotted).

respectively. From theoretical considerations, we expect the so-called betatron oscillations in these variables. This manifests itself as ellipses in the phase space plot of q_1 and p_1 variables. In Figure 1, we observe the expected betatron oscillations. We also see the thickening of the ellipses caused by nonlinearities present in the sextupoles.

6. Conclusions

To conclude, we described in detail a new symplectic integration algorithm based on polynomial map refactorization. The absence of poles and branch points in this method was highlighted. We studied the types of symplectic maps which give rise to polynomial actions on phase space variables. For three degrees of freedom, we obtained the refactorization of a given symplectic map in terms of polynomial maps. This refactorized map was then used to study long term stability of a complicated particle storage ring.

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