Canonical representations of sp(2*n*,*R*)

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In this paper a rather unconventional real basis for the real symplectic algebra sp(2n,R) is studied. This basis is valid for representations carried by homogeneous polynomials of the 2n phase-space variables. The utility of this basis for practical computations is demonstrated by giving a simple derivation of the second- and fourth-order indices of irreducible representations of sp(2n,R).

I. INTRODUCTION

In this paper we study irreducible representations (irreps) of the real symplectic Lie algebra sp(2n,R)[1] carried by homogeneous polynomials of the 2n phase-space variables $\{q_i,p_i\}$ $(i=1,2,...,n)$. These representations of sp(2n,R) are the physically interesting ones in classical mechanics.[2] Since these are representations carried by functions of canonically conjugate variables, we refer to them as canonical representations.

First, we define a real basis for sp(2n,R). Using this basis, irreps of sp(2n,R) carried by homogeneous polynomials of phase-space variables are obtained. Weight vectors corresponding to these irreps are also computed. Using these results, we finally give a simple derivation of the second- and fourth-order indices corresponding to these irreps.

II. A REAL BASIS FOR sp(2*n*,*R*)

In this section we obtain a real basis for sp(2n,R). Since we are interested in representations carried by functions of phase-space variables, it is convenient to define the operators constituting a basis for sp(2n,R) using these variables. This can be done using the concept of a Lie operator.[2] Let us denote the collection of 2n phase-space variables q_i, p_i $(i=1,2,...,n)$ by the symbol z. The Lie operator corresponding to a phase-space function $f(z)$ is denoted by $:f(z):$. It is defined by its action on a phase-space function $g(z)$ as shown below:

$$:f(z):g(z)=[f(z),g(z)]. \tag{2.1}$$

Here, $[f(z),g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$:

$$[f(z),g(z)]\equiv\sum_{i=1}^{3}\left(\frac{\partial f(z)}{\partial q_i}\frac{\partial g(z)}{\partial p_i}-\frac{\partial f(z)}{\partial p_i}\frac{\partial g(z)}{\partial q_i}\right). \tag{2.2}$$

We are now in a position to give a basis for sp(2n,R). Consider the following set of real operators:

$$\begin{aligned}&:C_{jk}:=:q_j p_k:,\\&:L_{jk}:=:q_j q_k:,\quad j\leqslant k,\\&:R_{jk}:=:p_j p_k:,\quad j\leqslant k,\end{aligned} \tag{2.3}$$

where the indices j and k range from 1 to n. We note that these are nothing but Lie operators corresponding to the set of all quadratic monomials in variables q_j and p_k. It can be shown that these operators constitute a real basis for the Lie algebra sp(2n,R).[3] As expected, there are $n(2n+1)$ elements in this basis. We will denote a general basis element by the symbol w_l.

The commutator of two Lie operators $:f:$ and $:g:$ can be shown[2] to satisfy the following relation:

$$\{:f:,:g:\}\equiv:f::g:-:g::f:=:[f,g]:. \tag{2.4}$$

Using this property, the basis elements are seen to satisfy the following commutation relations:

$$\begin{aligned}&\{:C_{jk}:,:C_{rs}:\}=:C_{rk}:\delta_{js}-:C_{js}:\delta_{kr},\\&\{:C_{jk}:,:L_{rs}:\}=-:L_{js}:\delta_{kr}-:L_{jr}:\delta_{ks},\\&\{:C_{jk}:,:R_{rs}:\}=:R_{ks}:\delta_{jr}+:R_{kr}:\delta_{js},\\&\{:L_{jk}:,:L_{rs}:\}=0,\\&\{:R_{jk}:,:R_{rs}:\}=0,\\&\{:L_{jk}:,:R_{rs}:\}=:C_{ks}:\delta_{jr}+:C_{kr}:\delta_{js}+:C_{js}:\delta_{kr}+:C_{jr}:\delta_{ks}.\end{aligned} \tag{2.5}$$

Here, the indices j, k, r, and s range from 1 to n.

The basis given in Eq. (2.3) is not the conventional basis used for sp(2n,R).[4] It does not contain a basis for the unitary algebra u(3) as a subset. However, this shortcoming is of no consequence when one studies only the symplectic algebra without any reference to its unitary subalgebras. In such cases the real basis defined in Eq. (2.3) offers several advantages due to its one-to-one correspondence with the set of quadratic monomials. These advantages will become apparent in the discussions that follow.

III. REPRESENTATION OF sp(2*n,R*)

In this section we first obtain the irreps of sp$(2n,R)$ carried by homogeneous polynomials of phase-space variables. Next, we compute the weight vectors corresponding to these irreps.

From the previous section it is clear that Lie operators corresponding to the set of all quadratic polynomials in z constitute the symplectic Lie algebra sp$(2n,R)$. Therefore, an N-dimensional representation of sp$(2n,R)$ is obtained by mapping each element (denoted in general by $:f_2:$) onto an $N\times N$ matrix $d(f_2)$ such that the following conditions are satisfied for all: $f_2:$, $:g_2:$ belonging to sp$(2n,R)$:

$$d(af_2+bg_2)=ad(f_2)+bd(g_2),\quad a,b\in\mathscr{R}, \tag{3.1}$$

$$d([f_2,g_2])=\{d(g_2),d(f_2)\}. \tag{3.2}$$

A representation d is said to be (completely) reducible if it can be converted into a block diagonal form using a similarity transformation. Otherwise, it is said to be irreducible.

Irreducible representations of sp$(2n,R)$ carried by homogeneous polynomials in z can be obtained in a straightforward fashion. Let $\mathscr{P}^{(m)}(z)$ denote the set of all homogeneous polynomials in z of degree m and $g_m(z)$ denote a general element belonging to this set. Consider the following property of the Lie operator $:f_2:$ [cf. Eq. (2.2)]:

$$:f_2:g_m=[f_2,g_m]=h_m, \tag{3.3}$$

where h_m is an element of $\mathscr{P}^{(m)}(z)$. That is, the elements $:f_2:$ of sp$(2n,R)$ preserve the degree of the polynomial on which they act. We therefore get the following relation:

$$:f_2:g_m=[f_2,g_m]\in\mathscr{P}^{(m)}\quad\forall g_m\in\mathscr{P}^{(m)}. \tag{3.4}$$

In other words, the set of all elements belonging to sp$(2n,R)$ leaves $\mathscr{P}^{(m)}$ invariant.

Let $\{P_\alpha^{(m)}\}$ be a basis for the set $\mathscr{P}^{(m)}$. Typically, we choose these basis elements to be the monomials of degree m in the $2n$ phase-space variables. The number of basis monomials $N(m)$ of degree m in the $2n$ phase-space variables is given by the relation[5,3]

$$N(m)=\binom{2n+m-1}{m}. \tag{3.5}$$

From Eq. (3.4) and the completeness of the set $\{P_\alpha^{(m)}\}$, we get the following relation:

$$:f_2:P_\alpha^{(m)}(z)=\sum_{\beta=1}^{N(m)} d^{(m)}(f_2)_\alpha^\beta P_\beta^{(m)}(z),$$
$$\alpha=1,2,\ldots,N(m), \tag{3.6}$$

where $d^{(m)}(f_2)_\alpha^\beta$ are coefficients multiplying the basis elements.

As we vary α from 1 to $N(m)$ in Eq. (3.6), the set of coefficients $d^{(m)}(f_2)_\alpha^\beta$ gives rise to an $N(m)\times N(m)$ matrix, $d^{(m)}(f_2)$, for each $:f_2:$ belonging to sp$(2n,R)$. It is easily verified that the set of such matrices (obtained by letting $:f_2:$ range over the entire Lie algebra) gives an $N(m)$-dimensional representation of sp$(2n,R)$. It can be shown[3] that these representations (for each m) are in fact irreducible.

Next, we obtain the weight vectors for irreps of sp$(2n,R)$ defined above. First, we need the Cartan subalgebra of sp$(2n,R)$. It is easily verified that the Cartan subalgebra of sp$(2n,R)$ is spanned by the following n elements:

$$H_1=:q_1p_1:,\quad H_2=:q_2p_2:,\quad \ldots,\quad H_n=:q_np_n:. \tag{3.7}$$

As expected, the rank of sp$(2n,R)$ is equal to n. The $N(m)$-dimensional irrep of H_i is given by Eq. (3.6).

To proceed further we need to choose the $N(m)$ basis elements $P_\alpha^{(m)}(z)$. We make the simplest choice. We choose $\{P_\alpha^{(m)}(z)\}$ to be the set of all mth degree monomials in the $2n$ phase-space variables. Therefore, the general element $P_\alpha^{(m)}(z)$ is given by the following relation:

$$P_\alpha^{(m)}(z)=q_1^{r_1}p_1^{r_2}\cdots q_n^{r_{2n-1}}p_n^{r_{2n}},$$
$$r_1+r_2+\cdots+r_{2n}=m. \tag{3.8}$$

Now, the advantage of using the rather unconventional basis given in Eq. (2.3) becomes apparent. In this basis the $N(m)$-dimensional irreps of the Cartan basis elements H_i are automatically diagonal! This is true for all the irreps defined by Eq. (3.6) Moreover, it is a simple matter to calculate the general weight vector. Using Eq. (3.8) we obtain the following results:

$$:q_ip_i:P_\alpha^{(m)}(z)=(r_{2i}-r_{2i-1})P_\alpha^{(m)}(z),\quad i=1,2,\ldots,n. \tag{3.9}$$

The weight vector $\lambda^{(\alpha)}$ is defined by the relation

$$:q_ip_i:P_\alpha^{(m)}(z)=\lambda_i^{(\alpha)}P_\alpha^{(m)}(z),\quad i=1,2,\ldots,n. \tag{3.10}$$

Comparing Eqs. (3.9) and (3.10) we obtain the result

$$\lambda^{(\alpha)}=(r_2-r_1,r_4-r_3,\ldots,r_{2n}-r_{2n-1}). \tag{3.11}$$

From Eq. (3.11) we see that the highest weight vector for the $N(m)$-dimensional irrep is given by

$(m,0,...,0)$. Thus our real basis is valid for the class of representations with the highest weight vector of the form $(m,0,...,0)$ for $m=1,2,3,...$.

IV. INDICES OF IRREDUCIBLE REPRESENTATIONS

Indices of irreducible representations[6] play an important part in representation theory. For example, they can be used to decompose direct products of representations and compute branching rules. In this section we illustrate the utility of the basis given in Eq. (2.3) by explicitly computing the second- and fourth-order indices of the irreps carried by homogeneous polynomials.

A. Second-order index

The second-order index is defined as follows:[6]

$$I_m^{(2)} = \sum_{\alpha=1}^{N(m)} [\lambda^{(\alpha)}]^2, \tag{4.1}$$

where

$$[\lambda^{(\alpha)}]^2 = \sum_{i=1}^{n} [\lambda_i^{(\alpha)}]^2. \tag{4.2}$$

Here, n is the rank of $\text{sp}(2n,R)$, $\lambda_i^{(\alpha)}$ is the ith component of the αth weight vector, and $N(m)$ is the dimension of the irrep carried by the mth degree homogeneous polynomials [cf. Eq. (3.5)]. The expression for $I_m^{(2)}$ in Eq. (4.1) can be simplified as follows. From symmetry considerations we have the following relation:

$$\sum_{\alpha=1}^{N(m)} [\lambda_i^{(\alpha)}]^2 = \sum_{\alpha=1}^{N(m)} [\lambda_j^{(\alpha)}]^2 \quad \forall i,j. \tag{4.3}$$

Hence, Eq. (4.1) can be rewritten as follows:

$$I_m^{(2)} = n \sum_{\alpha=1}^{N(m)} [\lambda_1^{(\alpha)}]^2. \tag{4.4}$$

Before proceeding further, we first establish the relation between the second-order index and the more familiar Dynkin index.[6] Using the fact that $d^{(m)}(q_i p_i)$ is diagonal, we obtain the relation [cf. Eq. (3.6)]

$$:q_1p_1:P_\alpha^{(m)}(z) = d^{(m)}(q_1p_1)_\alpha^\alpha P_\alpha^{(m)}(z)$$

$$\alpha = 1,2,...,N(m). \tag{4.5}$$

Comparing this with Eq. (3.10) we discover that Eq. (4.4) can be rewritten as follows:

$$I_m^{(2)} = n\,\text{Tr}[d^{(m)}(q_1p_1)d^{(m)}(q_1p_1)]. \tag{4.6}$$

However, the Dynkin index $\nu(m)$ is defined by the relation[7]

$$\nu(m) = \text{Tr}[d^{(m)}(w_\alpha)d^{(m)}(w_\beta)]/\text{Tr}[d^{(2)}(w_\alpha) \times d^{(2)}(w_\beta)]. \tag{4.7}$$

Since $\nu(m)$ is independent of the choice for w_α and w_β, we can take both to be equal to q_1p_1. Then we get the relation

$$I_m^{(2)} = \nu(m) I_2^{(2)}. \tag{4.8}$$

We now turn to the task of computing $I_m^{(2)}$. From Eq. (4.4) we see that we need to sum over squares of the first components of the weight vectors corresponding to all the monomials $P_\alpha^{(m)}(z)$ $(\alpha = 1,2,...N(m))$. Since $:q_1p_1:$ acts only on the q_1 and p_1 variables, we need to know only the q_1 and p_1 content of the basis monomials $P_\alpha^{(m)}(z)$. Hence, we write the general monomial as follows:

$$P_\alpha^{(m)}(z) = q_1^{m-k-l} p_1^l h_k(q_2,p_2,...,q_n,p_n), \tag{4.9}$$

where h_k stands for a monomial of degree k in the $2n-2$ variables $q_2,p_2,...,p_n$. The index k varies from 0 to m and the index l from 0 to $m-k$. The total number of monomials h_k of degree k is given by the relation [cf. Eq. (3.5)]

$$N'(k) = \binom{2n+k-3}{k}. \tag{4.10}$$

From Eqs. (3.11) and (4.9) we find the following expression for the first component $\lambda_1^{(\alpha)}$ of the weight vector corresponding to the monomial $P_\alpha^{(m)}(z)$:

$$\lambda_1^{(\alpha)} = -m + k + 2l. \tag{4.11}$$

Since all $N'(k)$ monomials with the same value of k have the same weight, we obtain the result

$$I_m^{(2)} = n \sum_{k=0}^{m} \sum_{l=0}^{m-k} \binom{2n+k-3}{k} (-m+k+2l)^2. \tag{4.12}$$

However, the following identity can be proved:

$$\sum_{l=0}^{m-k} (-m+k+2l)^2 = 2\binom{m-k+2}{m-k-1}. \tag{4.13}$$

Therefore, we obtain the relation

$$I_m^{(2)} = 2n \sum_{k=0}^{m-1} \binom{2n+k-3}{k} \binom{m-k+2}{m-k-1}. \tag{4.14}$$

After some manipulation, this can be shown to equal the following result:

$$I_m^{(2)}=2n\binom{2n+m}{2n+1}. \tag{4.15}$$

It is easily verified that this expression gives the correct second-order indices[8] for the irreps being studied (apart from an overall numerical factor).

From Eq. (4.8) the Dynkin index $\nu(m)$ is given by the relation

$$\nu(m)=\binom{2n+m}{2n+1}(2n+2)^{-1}. \tag{4.16}$$

Further, the eigenvalue $C_2(\Lambda)$ of the second-order Casimir operator can be calculated using the following relation:[7]

$$C_2(\Lambda)=N(2)\nu(m)/N(m), \tag{4.17}$$

where $N(2)$ is equal to the dimension of sp $(2n,R)$. We obtain the following result:

$$C_2(\Lambda)=(m^2+2nm)/4(n+1). \tag{4.18}$$

B. Fourth-order index

The fourth-order index is defined as follows:[6]

$$I_m^{(4)}=\sum_{\alpha=1}^{N(m)}\left[\sum_{i=1}^{n}(\lambda_i^{(\alpha)})^2\right]^2. \tag{4.19}$$

Using symmetry considerations this can be rewritten as follows:

$$I_m^{(4)}=n\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^4+n(n-1)\times\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^2(\lambda_2^{(\alpha)})^2. \tag{4.20}$$

Proceeding as before we find, after some manipulation, the following relation:

$$\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^4=\sum_{k=0}^{m-1}\binom{2n+k-3}{k}\binom{m-k+2}{3}\times[-8+12(m-k)+6(m-k^2]/5. \tag{4.21}$$

Evaluating the sums, we obtain the result

$$\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^4=2\binom{2n+m}{2n+1}(2n^2+6m^2+12mn-7n-3)/(2n+3)(n+1). \tag{4.22}$$

Similarly, we can derive the relation

$$\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^2(\lambda_2^{(\alpha)})^2=\sum_{k=0}^{m-1}\sum_{l=0}^{m-k}(-m+k+2l)^2\times\sum_{k'=0}^{k-1}\sum_{l'=0}^{k-k'}\binom{2n+k'-5}{k'}(-k+k'+2l')^2. \tag{4.23}$$

This can be simplified to the following expression:

$$\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^2(\lambda_2^{(\alpha)})^2=2\sum_{k=1}^{m-1}\binom{2n+k-2}{2n-1}\sum_{l=0}^{m-k}(-m+k+2l)^2. \tag{4.24}$$

Evaluating this double sum, we obtain the result

$$\sum_{\alpha=1}^{N(m)}(\lambda_1^{(\alpha)})^2(\lambda_2^{(\alpha)})^2=4\binom{2n+m+1}{2n+3}. \tag{4.25}$$

Substituting Eqs. (4.22) and (4.25) in Eq. (4.20), we finally get the following relation:

$$I_m^{(4)}=2n\binom{2n+m}{2n+1}(2mn^2+m^2n+10mn+5m^2-6n-2)/(2n+3)(n+1). \tag{4.26}$$

Again, it is easily verified that the above expression for $I_m^{(4)}$ gives correct results (up to an overall numerical factor). Comparing Eqs. (4.15) and (4.26) we also obtain the relation

$$I_m^{(4)}=I_m^{(2)}(2mn^2+m^2n+10mn+5m^2-6n-2)\times[(2n+3)(n+1)]^{-1}. \tag{4.27}$$

V. SUMMARY

In this paper we discuss a real basis for sp$(2n,R)$. In this basis the Cartan basis elements are automatically diagonal for all irreps carried by homogeneous polynomials of phase-space variables. This fact facilitates the computation of various quantities characterizing these irreps. As an illustration, we calculate the second- and fourth-order indices of these irreps.

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[1]The reader should be warned that there are several conflicting notations in use for denoting the various symplectic algebras. The notation $sp(2n)$ is often used to denote the real symplectic algebra $sp(2n,R)$. Sometimes $sp(n,R)$ is used to denote this algebra. In this paper we have adopted the least ambiguous set of notations found in the literature. The real symplectic algebra is denoted by $sp(2n,R)$; the complex symplectic algebra is denoted by $sp(2n,C)$; the so-called unitary symplectic algebra is denoted by $sp(2n)$.

[2]A. J. Dragt, in *Physics of High Energy Particle Accelerators*, edited by R. A. Carrigan, F. R. Huson, and M. Month, AIP Conference Proceedings No. 87 (American Institute of Physics, New York, 1982), p. 147; see also A. J. Dragt *et al.*, Ann. Rev. Nucl. Part. Sci. **38**, 455 (1988).
[3]G. Rangarajan, Ph. D. thesis, University of Maryland, 1990.
[4]H. Georgi, *Lie Algebras in Particle Physics* (Benjamin/Cummings, Reading, MA, 1982).
[5]A. Giorgilli Comp. Phys. Comm. **16**, 331 (1979).
[6]J. Patera, R. T. Sharp, and P. Winternitz, J. Math. Phys. **17**, 1972 (1976); J. Patera, R. T. Sharp, and P. Winternitz, J. Math. Phys. **18**, 1519 (1977).
[7]J. F. Cornwell, *Group Theory in Physics* (Academic, London, 1984), Vols. 1 and 2.
[8]W. G. McKay and J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Groups* (Marcel Dekker, New York, 1981).