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Jolt factorization of pendulum map

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Abstract. In this paper, we apply the symplectic integration method using jolt factorization described in an earlier paper to the symplectic map describing the nonlinear pendulum Hamiltonian. We compare results obtained with this method with those obtained using non-symplectic methods and demonstrate that our results are much better.

1. Introduction

In a previous paper [1] (henceforth referred to as paper I), we had described a general method for symplectic integration of nonlinear Hamiltonian systems using jolt factorization. A symplectic integration method explicitly preserves the Hamiltonian nature of the system during numerical integration. There has been a lot of activity recently in formulating symplectic integration algorithms [2–18]. In this paper, we consider an application of our symplectic integration method to the nonlinear pendulum Hamiltonian. This example was chosen since the pendulum is a prototypical nonlinear Hamiltonian system which has the advantage that analytical solutions are available facilitating easy comparisons. Further, non-symplectic integration methods can give wrong results even when applied to this simple problem. Figure 1 shows the results of numerically integrating the pendulum Hamiltonian using an eighth-order Taylor series map with a time step equal to 1 (a description of this map is given in section 3 in the paragraph following equation (3.16)). Comparison with the exact result (figure 2) shows that this non-symplectic integration method gives rise to spurious 'chaotic' behaviour where there is none. Such problems become accentuated when long-term integration is performed to study stability of Hamiltonian systems. This spurious behaviour in non-symplectic integration methods can be reduced by using a very short time step. However, this makes these methods so slow that they are impractical to use for long-term integration. We will show that our method gives much better results when applied to this problem even with this large time step (equal to 1). Finally, by dealing with a one degree-of-freedom system, the details of our symplectic integration method (described only in general terms in I) become clearer.

We start by describing the jolt factorization method for a two-dimensional phase space. In section 3, we apply this method to the pendulum map and compare our results both with analytical results and those obtained using a non-symplectic eighth-order Taylor series map. Our conclusions are summarized in the final section.

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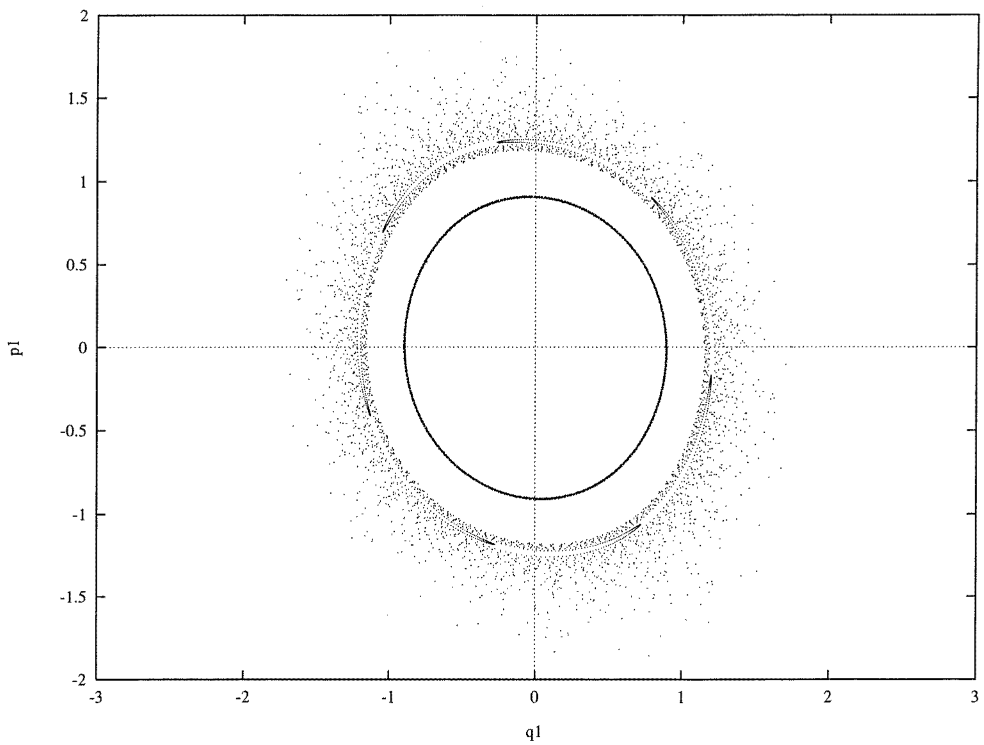

Figure 1. Eighth-order Taylor series approximation of pendulum map ($t = 1.0$). Numerical integration results using two sets of initial conditions are shown. $(q_1^{in}, p_1^{in}) = (0.9, 0.0)$ and $(1.75, 0.0)$.

2. Jolt factorization in a two-dimensional phase space

Consider a nonlinear Hamiltonian system in a two-dimensional phase space described by the Hamiltonian $H(q_1, p_1)$ where q_1 is the coordinate and p_1 is the momentum. We will collectively describe q_1 and p_1 by a single 2-vector $z = (q_1, p_1)$. The time evolution of this Hamiltonian system over time t can be represented by a symplectic map $\mathcal{M}$ [19] as follows

$$z^{(1)} = \mathcal{M} z^{(0)}. \tag{2.1}$$

Thus, $\mathcal{M}$ maps the initial phase space variables $z^{(0)}$ to their final values $z^{(1)}$ after time t. The symplectic map $\mathcal{M}$ can be factorized as shown below to order P [19]

$$\mathcal{M}_P = \hat{M} e^{:f_3:} e^{:f_4:} \ldots e^{:f_P:}. \tag{2.2}$$

We obtain a symplectic integration algorithm as follows. As described in I, we find another map $\mathcal{J}$ specified by the following product of $P+1$ jolt maps

$$\mathcal{J} = \hat{M} e^{:g_3^{(1)} + g_4^{(1)} + \cdots + g_P^{(1)}:} e^{:g_3^{(2)} + g_4^{(2)} + \cdots + g_P^{(2)}:} \ldots e^{:g_3^{(P+1)} + g_4^{(P+1)} + \cdots + g_P^{(P+1)}:} \tag{2.3}$$

such that this map agrees with $\mathcal{M}_P$ to order P. In the above equation, $g_n^{(i)}$'s are jolt polynomials given by the following relation

$$g_n^{(i)} = \beta_n^{(i)} \hat{R}_i q_1^n \qquad i = 1, 2, \ldots, P+1 \tag{2.4}$$

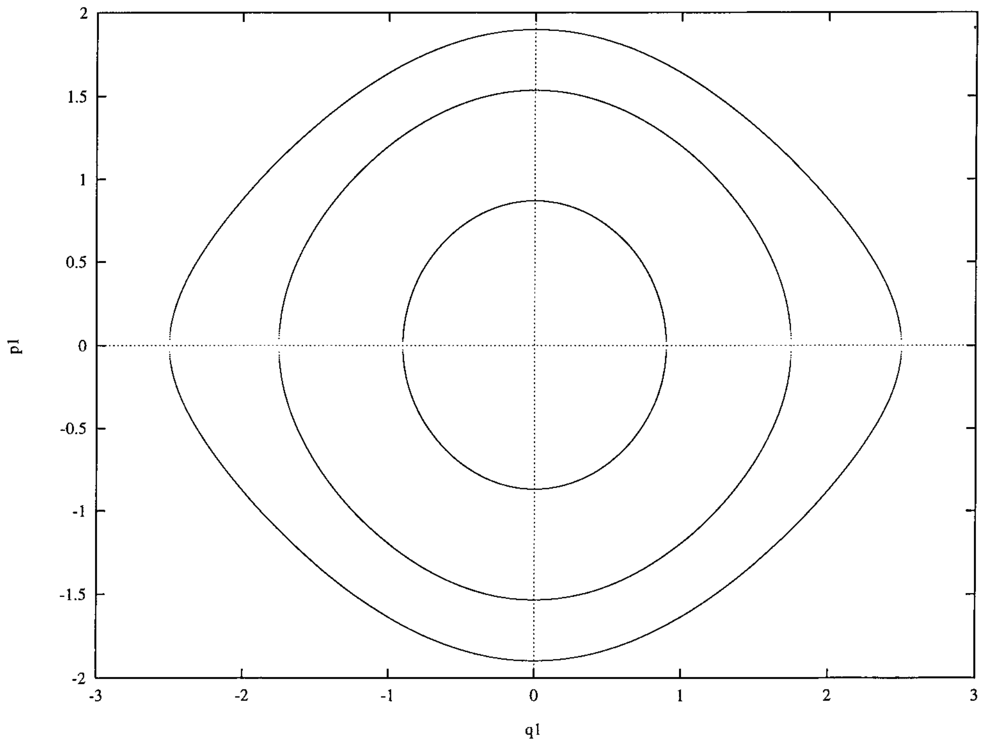

Figure 2. Exact pendulum map. Results using three sets of initial conditions are shown. $(q_1^{in}, p_1^{in}) = (0.9, 0.0)$, $(1.75, 0.0)$, and $(2.5, 0.0)$.

where $\beta_n^{(i)}$ are unknown coefficients to be determined by our algorithm. The matrices R_i belong to a subgroup of $\mathrm{Sp}(2, \mathbb{R})$ and $\hat{R}_i$ denotes the Lie transformation corresponding to these matrices:

$$\hat{R} z_i = R_{ij} z_j = (Rz)_i \qquad i = 1, 2. \tag{2.5}$$

From I, the problem of determining the jolt factored map $\mathcal{J}$ can be reduced to the problem given below.

Problem 1. Given an nth degree homogeneous polynomial f_n' in two phase space variables, find coefficients $\beta_n^{(i)}$'s such that the following conditions are satisfied

$$\sum_{i=1}^{P+1} \beta_n^{(i)} \hat{R}_i q_1^n = f_n' \tag{2.6}$$

$$\sum_{i=1}^{P+1} [\beta_n^{(i)}]^2 \qquad \text{is a minimum.} \tag{2.7}$$

Here the matrices R_i (belonging to a subgroup of $\mathrm{Sp}(2, \mathbb{R})$) satisfy the restriction

$$Q_i^{(P)} = \hat{R}_i q_1^P \ (i = 1, 2, \ldots, P + 1) \text{ form a linearly independent set of vectors.} \tag{2.8}$$

Further, f_n' is a sum of f_n and other nth order terms produced by concatenation of lower order terms through Baker–Campbell–Hausdorff (BCH) series [1].

It is thus convenient to go to the continuum limit of the above problem, solve it first in this limit, and then return to the discrete case. In this continuum limit, we obtain the following generalized problem.

Generalized problem 1. Given an nth degree homogeneous polynomial f_n', find the function $g(\theta)$ such that the following conditions are satisfied:

$$f_n' = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, g(\theta)\hat{R}(\theta)q_1^n \tag{2.9}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, g^2(\theta) \qquad \text{is a minimum.} \tag{2.10}$$

Since R belongs to $\mathrm{Sp}(2,\mathbb{R})$, ideally we should have integrated over this group. However, it is non-compact and therefore invariant integrations are not defined. Therefore, we have integrated over U(1), the largest compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$. The general group element has been labelled θ following standard convention. The action of $\hat{R}(\theta)$ on phase space variables can be computed and is given as follows

$$\hat{R}(\theta)q_1 = \cos\theta\ q_1 + \sin\theta p_1 \tag{2.11}$$

$$\hat{R}(\theta)p_1 = -\sin\theta\ q_1 + \cos\theta p_1. \tag{2.12}$$

In order to solve the problem, we introduce the so-called real resonance basis [19]

$$|kl\rangle_{\mathrm{R}} \equiv \mathrm{Re}\,|kl\rangle = \mathrm{Re}\,[(q_1 + \mathrm{i}p_1)^k(q_1 - \mathrm{i}p_1)^l] \tag{2.13}$$

$$|kl\rangle_{\mathrm{I}} \equiv \mathrm{Im}\,|kl\rangle = \mathrm{Im}\,[(q_1 + \mathrm{i}p_1)^k(q_1 - \mathrm{i}p_1)^l]. \tag{2.14}$$

They can be rewritten in terms of polar coordinates as follows

$$|kl\rangle_{\mathrm{R}} = r^{k+l}\cos(k-l)\phi \tag{2.15}$$

$$|kl\rangle_{\mathrm{I}} = r^{k+l}\sin(k-l)\phi \tag{2.16}$$

where

$$q_1 = r\cos\phi \qquad p_1 = r\sin\phi \qquad r = \sqrt{q_1^2 + p_1^2}. \tag{2.17}$$

It is easily seen that any nth degree homogeneous polynomial in variables q_1 and p_1 can be expressed in terms of these basis elements (with $k+l$ equal to n and k greater than or equal to l), i.e. these basis elements form a complete set.

Next, we consider the action of $\hat{R}(\theta)$ on the resonance basis elements. From equations (2.11) and (2.12), we have the relation

$$\hat{R}(\theta)|kl\rangle_{\mathrm{R}} = |kl\rangle_{\mathrm{R}}\cos(k-l)\theta + |kl\rangle_{\mathrm{I}}\sin(k-l)\theta \tag{2.18}$$

$$\hat{R}(\theta)|kl\rangle_{\mathrm{I}} = -|kl\rangle_{\mathrm{R}}\sin(k-l)\theta + |kl\rangle_{\mathrm{I}}\cos(k-l)\theta. \tag{2.19}$$

Therefore, $\hat{R}(\theta)$ is seen to rotate $|kl\rangle_{\mathrm{R}}$ and $|kl\rangle_{\mathrm{I}}$ into one another by an angle $(k-l)\theta$.

We now return to the task of solving the continuum limit problem outlined earlier (cf equations (2.9) and (2.10)). We will go back to the discrete case after obtaining this solution. First, we expand q_1^n in the real resonance basis

$$q_1^n = \sum_{\substack{k+l=n\\ k\geqslant l}} a_{kl}|kl\rangle_{\mathrm{R}} + b_{kl}|kl\rangle_{\mathrm{I}}. \tag{2.20}$$

The coefficients a_{kl} and b_{kl} are determined as follows. Using equation (2.17), we obtain the relation

$$q_1^n = r^n\cos^n\phi. \tag{2.21}$$

Expanding $\cos^n \phi$ we get [20]

$$q_1^n = \frac{r^n}{2^n}\left[\sum_{m=0}^{n/2-1} 2\binom{n}{m}\cos(n-2m)\phi + \binom{n}{n/2}\right] \qquad n \text{ even} \tag{2.22}$$

$$q_1^n = \frac{r^n}{2^{n-1}}\sum_{m=0}^{(n-1)/2} 2\binom{n}{m}\cos(n-2m)\phi \qquad n \text{ odd.} \tag{2.23}$$

Comparing these expressions with equation (2.20), we can read off the coefficients a_{kl} and b_{kl}:

$$a_{kl} = \frac{1}{2^{n-1}}\binom{n}{l} \qquad k+l=n,\ k>l \tag{2.24}$$

$$a_{kl} = \frac{1}{2^n}\binom{n}{n/2} \qquad k=l=n/2,\ n \text{ even} \tag{2.25}$$

$$b_{kl} = 0 \qquad \forall k,l. \tag{2.26}$$

Since b_{kl} is zero for all k, l, we will ignore it henceforth. Finally, using equations (2.18) and (2.19), we obtain the following result for the action of $\hat{R}(\theta)$ on q_1^n:

$$\hat{R}(\theta)q_1^n = \sum_{\substack{k+l=n\\ k\geqslant l}} [|kl\rangle_{\mathrm{R}} a_{kl}\cos(k-l)\theta + |kl\rangle_{\mathrm{I}} a_{kl}\sin(k-l)\theta]. \tag{2.27}$$

Next, we find functions $g_{kl}^{\mathrm{R}}(\theta)$ and $g_{kl}^{\mathrm{I}}(\theta)$ satisfying the following conditions:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, g_{kl}^{\mathrm{R}}(\theta)\hat{R}(\theta)q_1^n = |kl\rangle_{\mathrm{R}} \tag{2.28}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, g_{kl}^{\mathrm{I}}(\theta)\hat{R}(\theta)q_1^n = |kl\rangle_{\mathrm{I}}. \tag{2.29}$$

Using the orthonormality relations satisfied by the cosine and sine functions and equation (2.27), we immediately obtain the required solution

$$g_{kl}^{\mathrm{R}}(\theta) = \frac{\delta_{\mathrm{R}}}{a_{kl}^2} a_{kl}\cos(k-l)\theta \qquad k+l=n,\ k\geqslant l \tag{2.30}$$

$$g_{kl}^{\mathrm{I}}(\theta) = \frac{\delta_{\mathrm{I}}}{a_{kl}^2} a_{kl}\sin(k-l)\theta \qquad k+l=n,\ k\geqslant l \tag{2.31}$$

where

$$\delta_{\mathrm{R}} = 2 - \delta_{kl} \tag{2.32}$$

$$\delta_{\mathrm{I}} = 2 - 2\delta_{kl}. \tag{2.33}$$

Here δ_{kl} is the usual Kronecker delta. We also note that $g_{kl}^{\mathrm{R}}(\theta)$ and $g_{kl}^{\mathrm{I}}(\theta)$ are well defined since a_{kl} is non-zero for all valid k and l (cf equations (2.24) and (2.25)).

The homogeneous polynomials f_n' can also be expanded in the real resonance basis

$$f_n'(q_1, p_1) = \sum_{\substack{k+l=n\\ k\geqslant l}} [c_{kl}|kl\rangle_{\mathrm{R}} + d_{kl}|kl\rangle_{\mathrm{I}}]. \tag{2.34}$$

Since f_n' is known, this equation determines the values of c_{kl} and d_{kl}. Substituting equations (2.28) and (2.29) into the right-hand side of the above equation, we can immediately write down one solution of equation (2.9):

$$g(\theta) = \sum_{\substack{k+l=n\\ k\geqslant l}} [c_{kl}g_{kl}^{\mathrm{R}}(\theta) + d_{kl}g_{kl}^{\mathrm{I}}(\theta)]. \tag{2.35}$$

It is easily verified that this is indeed a correct solution by direct substitution into equation (2.9). It can be easily shown that it also satisfies equation (2.10).

We now go back to the discrete version of the problem (cf equations (2.6) and (2.7)) by making the following transformation

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \longrightarrow \frac{1}{K}\sum_{i=1}^{K}. \tag{2.36}$$

The discrete version of equation (2.9) is then given as follows

$$\frac{1}{K}\sum_{i=1}^{K} g(\theta_i)\hat{R}(\theta_i)q_1^n = f_n' \tag{2.37}$$

where $g(\theta)$ is given by equation (2.35). Comparing this expression with equation (2.6), we obtain the following relations

$$\beta_n^{(i)} = g(\theta_i)/K \qquad i = 1, 2, \ldots, K \tag{2.38}$$

$$\hat{R}_i = \hat{R}(\theta_i) \qquad i = 1, 2, \ldots, K. \tag{2.39}$$

If we choose the angles θ_i to be equally spaced over the interval $[0, 2\pi]$, the functions $\cos\theta_i$ and $\sin\theta_i$ still form an orthogonal set. Therefore, the solution $g(\theta)$ given in equation (2.35) is still the required solution. The only difference is that $g(\theta)$ is now evaluated only at the following K equally spaced points

$$\theta_i = (i-1)\frac{2\pi}{K} \qquad i = 1, 2, \ldots, K. \tag{2.40}$$

Substituting the expression of $\beta_i^{(n)}$ (cf equation (2.38)) into equation (2.4), we obtain the following jolt polynomials:

$$g_n^{(i)} = \frac{1}{K} g(\theta_i)\hat{R}(\theta_i)q_1^n \qquad i = 1, 2, \ldots, K. \tag{2.41}$$

Using these jolt polynomials in equation (2.4), we get the desired jolt factorization formula.

3. Jolt factorization of the pendulum map

The nonlinear pendulum is described by the following Hamiltonian H:

$$H(q_1, p_1) = \frac{p_1^2}{2} - \cos q_1 + 1. \tag{3.1}$$

To apply the method of jolt factorization, we need to represent this Hamiltonian in terms of a symplectic map $\mathcal{M}$. First, we expand $\cos q_1$ as a power series in q_1:

$$\cos q_1 = 1 - \frac{q_1^2}{2!} + \frac{q_1^4}{4!} - \frac{q_1^6}{6!} + \frac{q_1^8}{8!} - \cdots. \tag{3.2}$$

Substituting this into equation (3.1), we get the following expression for the Hamiltonian:

$$H(q_1, p_1) = \frac{p_1^2}{2!} + \frac{q_1^2}{2!} - \frac{q_1^4}{4!} + \frac{q_1^6}{6!} - \frac{q_1^8}{8!} + \cdots. \tag{3.3}$$

We now obtain the symplectic map $\mathcal{M}$ corresponding to this Hamiltonian (for unit time):

$$\mathcal{M} = \hat{M}\mathrm{e}^{:f_4:}\mathrm{e}^{:f_6:}\mathrm{e}^{:f_8:}\ldots. \tag{3.4}$$

Write the quadratic part of H (denoted by H_2) as follows

$$H_2 = \tfrac{1}{2}\sum_{a,b} S_{ab}z_a z_b. \tag{3.5}$$

Comparison with equation (3.3) gives

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

Since S is time independent, M is given by [21]

$$M = \exp[tJS] \tag{3.7}$$

where the starting time has been taken to be zero and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.8}$$

Explicitly evaluating the exponential in the expression for M we get

$$M = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{3.9}$$

For unit time this reduces to

$$M = \begin{pmatrix} 5.403 \times 10^{-1} & 8.415 \times 10^{-1} \\ -8.415 \times 10^{-1} & 5.403 \times 10^{-1} \end{pmatrix}. \tag{3.10}$$

Following the procedure in [21], it is trivially seen that all f_m's for m odd are zero since the corresponding H_m (which denotes the sum of terms in H of degree m in components of z) is zero. The polynomial f_4 is given by [21]

$$f_4 = -\int_0^t \mathrm{d}t'\, H_4(\hat{M}^{-1}z) \tag{3.11}$$

where M is given by equation (3.9) and $H_4 = -q_1^4/4!$ (cf equation (3.3)). Substituting these expressions into the above equation we obtain

$$f_4 = \frac{1}{24}\int_0^t \mathrm{d}t'\, (q_1 \cos t' - p_1 \sin t')^4. \tag{3.12}$$

Evaluating this integral for unit time

$$\begin{aligned} f_4 = {} & 2.411 \times 10^{-2} q_1^4 - 3.812 \times 10^{-2} q_1^3 p_1 + 3.716 \times 10^{-2} q_1^2 p_1^2 \\ & -2.089 \times 10^{-2} q_1 p_1^3 + 5.168 \times 10^{-3} p_1^4. \end{aligned} \tag{3.13}$$

Similarly, f_6 and f_8 can be evaluated to give

$$\begin{aligned} f_6 = {} & 1.200 \times 10^{-4} q_1^6 - 1.026 \times 10^{-3} q_1^5 p_1 + 1.390 \times 10^{-3} q_1^4 p_1^2 - 6.657 \times 10^{-4} q_1^3 p_1^3 \\ & -8.008 \times 10^{-5} q_1^2 p_1^4 + 1.748 \times 10^{-4} q_1 p_1^5 - 4.963 \times 10^{-5} p_1^6 \end{aligned} \tag{3.14}$$

$$\begin{aligned} f_8 = {} & 1.900 \times 10^{-5} q_1^8 - 1.069 \times 10^{-5} q_1^7 p_1 - 1.255 \times 10^{-4} q_1^6 p_1^2 + 3.172 \times 10^{-4} q_1^5 p_1^3 \\ & -3.530 \times 10^{-4} q_1^4 p_1^4 + 2.228 \times 10^{-4} q_1^3 p_1^3 - 8.331 \times 10^{-5} q_1^2 p_1^6 \\ & +1.778 \times 10^{-5} q_1 p_1^7 - 1.706 \times 10^{-6} p_1^8. \end{aligned} \tag{3.15}$$

Similar expressions can be obtained for higher order f_n's.

For simplicity, let us truncate the symplectic map representing the pendulum Hamiltonian at eighth order, i.e.

$$\mathcal{M}_8 = \hat{M} e^{:f_4:} e^{:f_6:} e^{:f_8:}. \tag{3.16}$$

One way of evaluating the action of this map on phase space coordinates would be to expand out the exponentials using the Taylor series expansion. Keeping terms to order 8, we obtain

$$\mathcal{M}_8 z_i = \hat{M}\left(1 + :f_4: + \frac{:f_4:^2}{2} + \frac{:f_4:^3}{6}\right)(1 + :f_6:)(1 + :f_8:) z_i \qquad i = 1, 2. \tag{3.17}$$

This gives the eighth-order Taylor series map. Figure 1 displays the results obtained using this map. Since the map is non-symplectic, it gives rise to spurious ‘chaotic’ behaviour as observed earlier.

We now derive the symplectic jolt factorization of this map and find that it gives much better results. First, we fix the number of jolt maps and the matrices R_i. Since the maximum order P of the map is equal to 8, the number K of jolt maps required is 9 (cf the discussion before equation (2.6)). We choose the nine corresponding angles θ_i according to equation (2.40) and the nine matrices R_i according to equation (2.39).

Next, we consider the fourth-order term. Expanding q_1^4 in the real resonance basis, we obtain the relation (cf equations (2.20), (2.24) and (2.25))

$$q_1^4 = a_{22}|22\rangle_{\mathrm{R}} + a_{31}|31\rangle_{\mathrm{R}} + a_{40}|40\rangle_{\mathrm{R}} \tag{3.18}$$

where

$$a_{22} = \tfrac{3}{8} \qquad a_{31} = \tfrac{1}{2} \qquad a_{40} = \tfrac{1}{8}. \tag{3.19}$$

The functions $g^{R,I}(\theta)$ are found to be given by the following expressions (cf equation (2.30) and (2.31))

$$g_{22}^{\mathrm{R}}(\theta) = \frac{1}{a_{22}} = \frac{8}{3} \tag{3.20}$$

$$g_{31}^{\mathrm{R}}(\theta) = \frac{2}{a_{31}} \cos 2\theta = 4\cos 2\theta \tag{3.21}$$

$$g_{31}^{\mathrm{I}}(\theta) = \frac{2}{a_{31}} \sin 2\theta = 4\sin 2\theta \tag{3.22}$$

$$g_{40}^{\mathrm{R}}(\theta) = \frac{2}{a_{40}} \cos 4\theta = 16\cos 4\theta \tag{3.23}$$

$$g_{40}^{\mathrm{I}}(\theta) = \frac{2}{a_{40}} \sin 4\theta = 16\sin 4\theta. \tag{3.24}$$

Since f_4 is the lowest order nonlinear term in $\mathcal{M}_8$, f_4' is equal to f_4 (i.e. no extra terms have yet been produced). Expanding f_4 in the real resonance basis, we get the following relation

$$f_4(q_1, p_1) = c_{22}|22\rangle_{\mathrm{R}} + c_{31}|31\rangle_{\mathrm{R}} + d_{31}|31\rangle_{\mathrm{I}} + c_{40}|40\rangle_{\mathrm{R}} + d_{40}|40\rangle_{\mathrm{I}} \tag{3.25}$$

where

$$c_{22} = 1.563 \times 10^{-2} \qquad c_{31} = 9.472 \times 10^{-3} \qquad c_{40} = -9.854 \times 10^{-4} \tag{3.26}$$

$$d_{31} = -1.475 \times 10^{-2} \qquad d_{40} = -2.153 \times 10^{-3}. \tag{3.27}$$

Next, from equation (2.35), the following expression is obtained for $g(\theta)$

$$g(\theta) = c_{22}g_{22}^{\mathrm{R}}(\theta) + c_{31}g_{31}^{\mathrm{R}}(\theta) + d_{31}g_{31}^{\mathrm{I}}(\theta) + c_{40}g_{40}^{\mathrm{R}}(\theta) + d_{40}g_{40}^{\mathrm{I}}(\theta) \tag{3.28}$$

where the coefficients are given by equations (3.26) and (3.27) while the functions $g^{R,I}(\theta)$ are given by equations (3.20)–(3.24). Finally, from equation (2.38), we obtain the coefficients $\beta_i^{(4)}$:

$$\beta_i^{(4)} = g(\theta_i)/9 \qquad i = 1, 2, \ldots, 9. \tag{3.29}$$

Evaluating this, we find the following results

$$\begin{aligned} &\beta_4^{(1)} = 7.087 \times 10^{-3} \qquad \beta_4^{(2)} = -7.590 \times 10^{-4} \qquad \beta_4^{(3)} = -4.501 \times 10^{-4} \\ &\beta_4^{(4)} = 5.764 \times 10^{-3} \qquad \beta_4^{(5)} = 1.553 \times 10^{-2} \qquad \beta_4^{(6)} = -4.337 \times 10^{-4} \\ &\beta_4^{(7)} = 1.038 \times 10^{-3} \qquad \beta_4^{(8)} = -8.864 \times 10^{-4} \qquad \beta_4^{(9)} = 1.477 \times 10^{-2}. \end{aligned} \tag{3.30}$$

Using a similar procedure, we can determine the coefficients $\beta_6^{(i)}$ and $\beta_8^{(i)}$ (for $i = 1, 2, \ldots, 9$) corresponding to the sixth- and eighth-order terms respectively:

$$\begin{array}{lll} \beta_6^{(1)} = -9.946 \times 10^{-5} & \beta_6^{(2)} = 3.725 \times 10^{-4} & \beta_6^{(3)} = 5.282 \times 10^{-5} \\ \beta_6^{(4)} = -7.000 \times 10^{-4} & \beta_6^{(5)} = 1.420 \times 10^{-3} & \beta_6^{(6)} = -3.345 \times 10^{-4} \\ \beta_6^{(7)} = -2.661 \times 10^{-4} & \beta_6^{(8)} = 2.727 \times 10^{-4} & \beta_6^{(9)} = 3.845 \times 10^{-4} \end{array} \tag{3.31}$$

and

$$\begin{array}{lll} \beta_8^{(1)} = 2.864 \times 10^{-5} & \beta_8^{(2)} = 7.757 \times 10^{-6} & \beta_8^{(3)} = 1.471 \times 10^{-5} \\ \beta_8^{(4)} = 5.516 \times 10^{-5} & \beta_8^{(5)} = -3.416 \times 10^{-7} & \beta_8^{(6)} = 2.061 \times 10^{-5} \\ \beta_8^{(7)} = -1.663 \times 10^{-5} & \beta_8^{(8)} = -3.652 \times 10^{-5} & \beta_8^{(9)} = 2.830 \times 10^{-5}. \end{array} \tag{3.32}$$

Putting everything together, the eighth symplectic map $\mathcal{M}_8$ (cf equation (3.16)) representing the pendulum Hamiltonian can be approximated by the following product of nine jolt maps

$$\mathcal{J} = \hat{M} e^{:g^{(1)}:} e^{:g^{(2)}:} \ldots e^{:g^{(9)}:} \tag{3.33}$$

where (cf equation (2.4))

$$g^{(i)} = \hat{R}_i [\beta_4^{(i)} q_1^4 + \beta_6^{(i)} q_1^6 + \beta_8^{(i)} q_1^8] \qquad i = 1, 2, \ldots, 9. \tag{3.34}$$

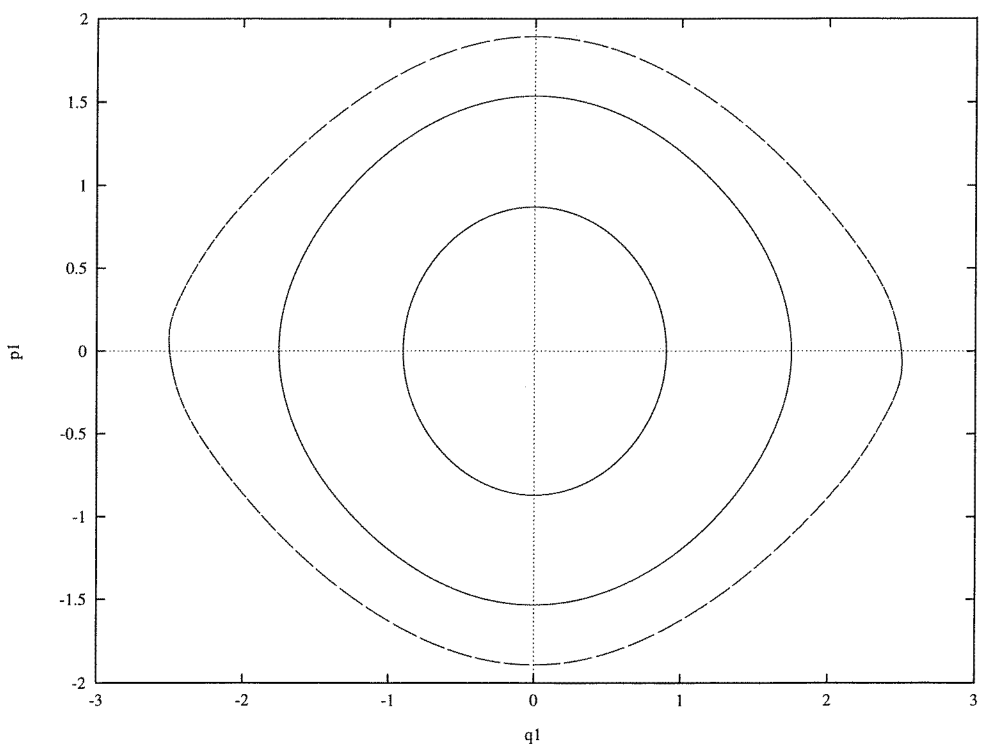

Figure 3. Eighth-order jolt factored pendulum map ($t = 1.0$). Numerical integration results using three sets of initial conditions are shown. $(q_1^{in}, p_1^{in}) = (0.9, 0.0)$, $(1.75, 0.0)$, and $(2.5, 0.0)$. The results agree very well with those obtained using the exact map (cf figure 2).

Here, the coefficients $\beta_4^{(i)}$, $\beta_6^{(i)}$, and $\beta_8^{(i)}$ are given by equations (3.30), (3.31) and (3.32) respectively. The matrices R_i are given by equation (2.39) and the angles θ_i are given by equation (2.40).

We can now numerically integrate the pendulum Hamiltonian using the map $\mathcal{J}$ given in equation (3.33). The results, for a variety of initial conditions, are shown in figure 3. These results should be compared with the exact results (valid for all times) given in figure 2. We see that the jolt factored map gives much better results than the truncated Taylor series map (figure 1).

4. Conclusions

In this paper, we applied the jolt factorization technique of symplectic integration to the pendulum map. We obtained an explicit formula for the jolt factored map (to order 8). This was used to numerically integrate the pendulum map. Numerical results agreed quite well with the exact results and were much better than the results obtained using the non-symplectic Taylor series method.

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