First Passage Distributions for Long Memory Processes

Govindan Rangarajan[1] and Mingzhou Ding[2]

[1] Department of Mathematics and Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India
[2] Centre for Complex Systems and Brain Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

Abstract. We study the distribution of first passage time for Levy type anomalous diffusion. A fractional Fokker-Planck equation framework is introduced. For the zero drift case, using fractional calculus an explicit analytic solution for the first passage time density function in terms of Fox or H-functions is given. The asymptotic behaviour of the density function is discussed. For the nonzero drift case, we obtain an expression for the Laplace transform of the first passage time density function, from which the mean first passage time and variance are derived.

1 Introduction

Consider a stochastic process $X(t)$ with $X(0) = 0$. The first passage time (FPT) T to the point $X = a$ is defined as [1]

$$T = \inf\{t : X(t) = a\}.$$

For ordinary diffusion modeled by Brownian motion, where $\mathrm{Var}(X(t)) = 2Kt$, the exact density function of T is known and given by

$$f_\mu(t) = \frac{a}{(4\pi K t^3)^{1/2}} e^{-(a-\mu t)^2/4Kt}, \tag{1}$$

where μ denotes the drift, K is the diffusion coefficient and we have assumed $a > 0$ for ease of expression. Equation (1) is often referred to as the inverse Gaussian distribution and was first obtained by Schrodinger [2] and by Smoluchowski [3] in 1915. The zero drift case of $\mu = 0$ was considered earlier by Bachelier [4] around 1900 in the context of financial analysis. (See Seshadri [5] for an interesting account of their work and a general historical survey.)

The FPT problem finds applications in many areas of science and engineering [6,7,8,9]. A sampling of these applications is listed below:

- probability theory (study of Wiener process, fractional Brownian motion etc.)
- statistical physics (study of anomalous diffusion)
- neuroscience (analysis of neuron firing models)
- civil and mechanical engineering (analysis of structural failure)
- chemical physics (study of noise assisted potential barrier crossings)
- hydrology (optimal design of dams)

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- financial mathematics (analysis of circuit breakers)
- imaging (study of image blurring due to hand jitter)

In this chapter we consider the FPT problem for a class of long memory processes. In particular, we study non Gaussian and non Markovian stochastic processes referred to as anomalous diffusions of the Levy type [10,11,12] where $\mathrm{Var}(X(t)) \sim t^{\gamma}$, $0 < \gamma < 2$, for large t. Generalizing earlier work for the zero drift case [13], our first result is the formulation of a fractional Fokker-Planck equation (FFPE) which describes Levy type anomalous diffusion with nonzero drift. Specializing to the zero drift case, we solve the FFPE under suitable initial and boundary value conditions using fractional calculus to obtain the FPT density function in terms of Fox or H-functions. We further prove that it is a valid probability density. For the nonzero drift case, we obtain the Laplace transform of the FPT density function. For $0 < \gamma < 1$, the Laplace transform is shown not to satisfy the completely monotone conditions [14] required for the Laplace transform of a valid probability density. Restricting ourselves to $1 \leq \gamma < 2$ we derive the mean and variance of the FPT density function. We show that for $\gamma = 1$, which corresponds to the ordinary diffusion, the inverse Laplace transform can be carried out explicitly and the result is (1). Finally, using properties of the H-functions, we obtain an asymptotic power law expression for the zero drift FPT density function.

2 Fractional Fokker-Planck Equation for Levy Type Anomalous Diffusion with Drift

Definition 1. Anomalous diffusion $X_{\gamma}(t)$ is a diffusive process with diffusion parameter γ $(0 < \gamma < 2)$ where the mean $\mathbf{E}(X_{\gamma}(t)) = 0$ and the mean square displacement

$$\mathrm{Var}(X_{\gamma}(t)) \sim t^{\gamma} \quad \text{for large } t.$$

For $\gamma = 1$ we obtain ordinary diffusion. Subdiffusion corresponds to $0 < \gamma < 1$ and superdiffusion corresponds to $1 < \gamma < 2$.

Anomalous diffusion of the Levy type is a class of non Gaussian and non Markovian processes founded on the continuous time random walk (CTRW) where the waiting time comes from a renewal process and obeys certain power law distribution [11]. Let $\xi(y,u)$ denote the joint probability density between the jump size y and the waiting time u. It can be shown that, depending on the specific form of $\xi(y,u)$, the CTRW can produce both subdiffusive and superdiffusive processes as well as ordinary diffusion [11]. For example, consider

$$\xi(y,u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-y^2/2\sigma^2] \frac{(\alpha-1)/\tau}{(1+u/\tau)^{\alpha}}, \tag{2}$$

where y and u are independent with y being a Gaussian variable and u a Levy stable variable [15]. For $1 < \alpha < 2$, the corresponding CTRW is characterized by a subdiffusive process with $\gamma = \alpha - 1$, and for $\alpha \geq 2$, one gets ordinary diffusion with $\gamma = 1$. If, on the other hand, y and u are coupled through

$$\xi(y, u) = \frac{1}{2}\delta(u/\tau - |y|/\sigma)\frac{(\beta - 1)/\tau}{(1 + u/\tau)^{\beta}},$$

where $2 < \beta < 3$ and $\delta(\cdot)$ is the Dirac delta function, the CTRW describes a superdiffusive process with $\gamma = 4 - \beta$.

Definition 2. Levy type anomalous diffusion $X_\gamma(t)$ is the non-Gaussian stochastic process obtained by taking the generalized diffusion limit of the above CTRW. This limit is defined as: $\sigma^2 \to 0$, $\tau \to 0$ such that $K = \sigma^2/2\Gamma(1-\gamma)\tau^\gamma$ is a constant for a subdiffusive process and $K = (2-\gamma)\Gamma(\gamma - 1)\sigma^2/2\tau^\gamma$ is a constant for a superdiffusive process. Here K is called the generalized diffusion constant.

Definition 3. Anomalous diffusion with drift μ is defined by

$$X_{\gamma,\mu}(t) = \mu t + X_\gamma(t).$$

Let the probability density function of a Levy type anomalous diffusive process with drift μ be denoted $p(x, t)$. Recently Metzler et al. [13] have formulated a fractional Fokker-Planck equation (FFPE), based on the renewal equation, which describes zero drift Levy type anomalous diffusive processes with $0 < \gamma \leq 1$. Generalizing this we obtain the following FFPE for the evolution of $p(x, t)$

$$p(x, t) - p(x, 0) = K \, {}_0D_t^{-\gamma} \frac{\partial^2}{\partial x^2} p(x, t) - \mu \, {}_0D_t^{-1} \frac{\partial}{\partial x} p(x, t), \tag{3}$$

where $0 < \gamma < 2$ and K is the generalized diffusion constant defined above. Here the Riemann-Liouville fractional integral operator ${}_0D_t^{-\gamma}$ is defined as [16,17]

$${}_0D_t^{-\gamma} p(x, t) = \frac{1}{\Gamma(\gamma)} \int_0^t dt' \; (t - t')^{\gamma - 1} p(x, t'), \quad \gamma > 0,$$

with $\Gamma(z)$ being the gamma function [18]. The integration kernel on the right hand side accounts for the non Markovian memory effect stemming from the nonexponential waiting time distribution. From the above FFPE it is easily seen that

$$\mathbf{E}(X_{\gamma,\mu}(t)) = \mu t; \quad \mathrm{Var}(X_{\gamma,\mu}(t)) = 2Kt^\gamma/\Gamma(1 + \gamma).$$

This demonstrates that our FFPE produces the correct time dependence for the mean and the variance.

To obtain the first passage time density function, we first solve (3) with the following initial and boundary value conditions:

$$p(\infty, t) = p(0, t) = 0 \;\; (t > 0); \quad p(x, 0) = \delta(x - a), \tag{4}$$

where $x = a$ is the starting point of the diffusive process, containing the initial concentration of the distribution. We note that the above conditions are slightly different from the standard ones [19] used in such problems where $x = 0$ is the starting point but both sets of conditions are equivalent. The present formulation makes the subsequent derivation less cumbersome. Once $p(x,t)$ is known, the first passage time density function $f_{\gamma,\mu}(t)$ is given by [19]:

$$f_{\gamma,\mu}(t) = -\frac{d}{dt}\int_0^\infty p(x,t)dx. \tag{5}$$

3 FPT Density Function for Levy Type Anomalous Diffusion with Zero Drift

For a Levy type anomalous diffusive process with zero drift, the fractional Fokker-Planck equation for $p(x,t)$ reduces to

$$p(x,t) - p(x,0) = K\ {}_0D_t^{-\gamma}\frac{\partial^2}{\partial x^2}p(x,t). \tag{6}$$

We now state and prove

Theorem 1. *For Levy type anomalous diffusion with zero drift ($\mu = 0$) described by (6), the FPT density function is given by*

$$f_{\gamma,0}(t) = \frac{a\gamma}{2K^{1/2}t^{(2+\gamma)/2}}H_{1,1}^{1,0}\left(\frac{a}{(Kt^\gamma)^{1/2}}\middle|\begin{matrix}(1-\gamma/2,\gamma/2)\\(0,1)\end{matrix}\right), \quad a,t>0, \tag{7}$$

where $H(\cdot)$ denotes the Fox or H-function [20,21,22] to be described below.

Proof. Taking the Laplace transform of (6) with respect to time we get

$$q(x,s) - \frac{p(x,0)}{s} = \frac{K}{s^\gamma}\frac{\partial^2}{\partial x^2}q(x,s),$$

where $q(x,s)$ is the Laplace transform of $p(x,t)$ with respect to time. Here we have also applied the result [17] that the Laplace transform of ${}_0D_t^{-\gamma}p(x,t)$ is $q(x,s)/s^\gamma$. Using the initial condition $p(x,0) = \delta(x-a)$, the above equation can be rewritten as

$$\frac{\partial^2}{\partial x^2}q(x,s) - \frac{s^\gamma}{K}q(x,s) = -\frac{s^{\gamma-1}}{K}\delta(x-a).$$

The general solution of this equation satisfying all the boundary and initial conditions [cf. (4)] is given by

$$q(x,s) = \frac{s^{\gamma/2-1}}{2\sqrt{K}}\left[\exp(-s^{\gamma/2}|x-a|/\sqrt{K}) - \exp(-s^{\gamma/2}(x+a)/\sqrt{K})\right].$$

The inverse Laplace transform of $s^{\gamma/2-1}\exp(-|x|s^{\gamma/2})$ is known [23]. Writing the result using the H-function (which is more convenient for our purpose) we get

$$p(x,t) = \frac{1}{2(Kt^\gamma)^{1/2}} H^{1,0}_{1,1}\left(\frac{|x-a|}{(Kt^\gamma)^{1/2}} \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right) - \frac{1}{2(Kt^\gamma)^{1/2}} H^{1,0}_{1,1}\left(\frac{x+a}{(Kt^\gamma)^{1/2}} \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right). \tag{8}$$

Here, the Fox or H-function [20,21,22] has the following alternating power series expansion:

$$H^{m,n}_{p,q}\left(z \middle| \begin{matrix} (a_j,A_j)_{j=1,\dots,p} \\ (b_j,B_j)_{j=1,\dots,q} \end{matrix} \right) = \sum_{l=1}^{m} \sum_{k=0}^{\infty} \frac{(-1)^k z^{s_{lk}}}{k! B_l} \times \frac{\prod_{j=1,j\neq l}^{m} \Gamma(b_j - B_j s_{lk}) \prod_{r=1}^{n} \Gamma(1-a_r+A_r s_{lk})}{\prod_{u=m+1}^{q} \Gamma(1-b_u+B_u s_{lk}) \prod_{v=n+1}^{p} \Gamma(a_v - A_v s_{lk})}, \tag{9}$$

where $s_{lk} = (b_l + k)/B_l$ and an empty product is interpreted as unity. Further, m, n, p, q are nonnegative integers such that $0 \le n \le p$, $1 \le m \le q$; A_j, B_j are positive numbers; a_j, b_j can be complex numbers. For further discussions of the H-function, see Mathai [21]. A few key properties of H-functions are summarized in the Appendix.

Substituting (8) into (5) we have

$$f_{\gamma,0}(t) = -\frac{d}{dt}\left[\frac{1}{2(Kt^\gamma)^{1/2}} \int_0^\infty dx\; H^{1,0}_{1,1}\left(\frac{|x-a|}{(Kt^\gamma)^{1/2}} \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right) \right] + \frac{d}{dt}\left[\frac{1}{2(Kt^\gamma)^{1/2}} \int_0^\infty dx\; H^{1,0}_{1,1}\left(\frac{x+a}{(Kt^\gamma)^{1/2}} \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right) \right].$$

Defining $z = (x-a)/(Kt^\gamma)^{1/2}$, $z' = (x+a)/(Kt^\gamma)^{1/2}$, we obtain

$$\begin{aligned} f_{\gamma,0}(t) &= -\frac{d}{dt} \int_{-a/(Kt^\gamma)^{1/2}}^{\infty} dz\; H^{1,0}_{1,1}\left(|z| \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right) \\ &\quad + \frac{d}{dt} \int_{a/(Kt^\gamma)^{1/2}}^{\infty} dz'\; H^{1,0}_{1,1}\left(z' \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right) \\ &= \frac{a\gamma}{2K^{1/2} t^{(2+\gamma)/2}} H^{1,0}_{1,1}\left(\frac{a}{(Kt^\gamma)^{1/2}} \middle| \begin{matrix} (1-\gamma/2,\gamma/2) \\ (0,1) \end{matrix} \right). \end{aligned}$$

This completes the proof. This result was obtained by us in earlier papers [24,25].

The above theoretical prediction for the full FPT density function were verified by numerically simulating the underlying CTRW process characterized by the probability density function $\xi(y,u)$ [cf. (2)]. The FPT density function obtained theoretically from Eq. (7) was compared with the FPT density function obtained numerically using 10 million realizations of the underlying stochastic process. We observed that the numerical simulation was in excellent agreement with the theoretical prediction.

Note that for ordinary diffusion ($\gamma = 1$), the expression for $f_{\gamma,0}(t)$ in (7) reduces to

$$f_{1,0}(t) = \frac{a}{2\sqrt{Kt^3}} \sum_{k=0}^{\infty} \frac{(-a/\sqrt{Kt})^k}{k!\Gamma(1/2 - k/2)}, \tag{10}$$

where we have used the alternating series expansion for the H-function given in (9). But [18]

$$\begin{aligned} \frac{1}{\Gamma(1/2 - k/2)} &= 0, \quad k \text{ odd}, \\ &= \frac{k!}{(-4)^{k/2}(k/2)!\sqrt{\pi}}, \quad k \text{ even}. \end{aligned}$$

Substituting this in (10) and letting $n = k/2$ we get

$$f_{1,0}(t) = \frac{a}{2\sqrt{Kt^3}} \sum_{n=0}^{\infty} \frac{(-a/\sqrt{Kt})^{2n}(2n)!}{(2n)!(-4)^n n!\sqrt{\pi}}.$$

Simplifying this we get the standard result [which agrees with (1) for $\mu = 0$]

$$\begin{aligned} f_{1,0}(t) &= \frac{a}{\sqrt{4\pi Kt^3}} \sum_{n=0}^{\infty} \frac{(-a^2/4\sqrt{Kt})^n}{n!} \\ &= \frac{a}{\sqrt{4\pi Kt^3}} \exp[-a^2/4Kt]. \end{aligned}$$

We next prove the following result.

Theorem 2. *$f_{\gamma,0}(t)$ is a proper probability density function.*

Proof. We know that $f_{\gamma,0}(t)$ is a proper probability density if and only if its Laplace transform $\phi_{\gamma,0}(s)$ satisfies the completely monotone condition [14], i.e.,

$$(-1)^n \frac{d^n \phi_{\gamma,0}(s)}{ds^n} \geq 0 \text{ for } s > 0, \ n = 1, 2, \ldots \quad \text{and } \phi_{\gamma,0}(0) = 1.$$

In our case,

$$\phi_{\gamma,0}(s) = e^{g(s)}, \tag{11}$$

where

$$g(s) = \frac{-a}{\sqrt{K}} s^{\gamma/2}. \tag{12}$$

To prove that $\phi_{\gamma,0}(s)$ is completely monotone, consider the following Lemma.

Lemma 2. *For $n \geq 1$, $(-1)^n \frac{d^n e^{g(s)}}{ds^n} \geq 0$ if $g^{(n)}(s)$ (the nth derivative of $g(s)$ with respect to s) is non-positive for n odd and non-negative for n even.*

Proof. We have

$$\frac{d^n e^{g(s)}}{ds^n} = \sum_{i=1}^{m} c_i [g^{(1)}]^{r_{i,1}} [g^{(2)}]^{r_{i,2}} \cdots [g^{(n)}]^{r_{i,n}} e^{g(s)},$$

where $r_{i,1}, r_{i,2}, \ldots, r_{i,n} \geq 0$, $c_i > 0$ and $r_{i,1} + r_{i,2} + \cdots + r_{i,n} \leq n$ for $i = 1, 2, \ldots, m$. Using standard differentiation rules from calculus, it is easily seen that $r_{i,1} + r_{i,3} + \cdots + r_{i,n}$ is odd for n odd and $r_{i,1} + r_{i,3} + \cdots + r_{i,n-1}$ is even for n even (where $i = 1, 2, \ldots, m$). This immediately gives us the required result thus completing the proof of Lemma 1.

Returning to the proof of Theorem 2, we have [cf. (12)]

$$g^{(n)}(s) = \frac{-a}{\sqrt{K}} \left(\frac{\gamma}{2}\right) \left(\frac{\gamma}{2} - 1\right) \cdots \left(\frac{\gamma}{2} - n + 1\right) s^{\gamma/2 - n}.$$

Since $0 < \gamma < 2$ and $s > 0$, $g^{(n)}(s)$ is non-positive for n odd and non-negative for n even. Combining this result with Lemma 1 and (11), we see that $(-1)^n \frac{d^n \phi_{\gamma,0}(s)}{ds^n} \geq 0$ for $n \geq 1$. From (11) we also see that $\phi_{\gamma,0}(0) = 1$. This completes the proof of Theorem 2.

Finally, we consider the asymptotic behaviour of the FPT density function for large values of t. Refer to (7). Let $z = a/(Kt^\gamma)^{1/2}$. It is known [21,27] that, for small z, $H_{1,1}^{1,0}(z) \sim |z|^{b_1/B_1} = 1$, since $b_1 = 0$ and $B_1 = 1$. Therefore, the FPT distribution $f(t)$, for large t, is characterized by the power law relation

$$f(t) \sim t^{-1-\gamma/2}, \quad t \to \infty, \tag{13}$$

which becomes the well known $-3/2$ scaling law for the ordinary Brownian motion. This power law behaviour has been observed earlier by Balakrishnan [28] for subdiffusive processes ($0 < \gamma < 1$) using a different method.

4 Laplace Transform of FPT Density Function for Levy Type Anomalous Diffusion with Drift

Theorem 3. *Given a Levy type anomalous diffusion with drift μ described by (3), the Laplace transform $\phi_{\gamma,\mu}(s)$ of its first passage time density function $f_{\gamma,\mu}(t)$ is given by*

$$\phi_{\gamma,\mu}(s) = \exp\left[-\frac{a\mu s^{\gamma-1}}{2K} - a\sqrt{\frac{\mu^2 s^{2\gamma-2}}{4K^2} + \frac{s^\gamma}{K}}\right]. \tag{14}$$

Proof. Taking the Laplace transform of Eq. (3) with respect to time we get

$$q(x,s) - \frac{p(x,0)}{s} = \frac{K}{s^\gamma} \frac{\partial^2}{\partial x^2} q(x,s) - \frac{\mu}{s} \frac{\partial}{\partial x} q(x,s),$$

where $q(x,s)$ is the Laplace transform of $p(x,t)$. Here we have again applied the result [17] that the Laplace transform of ${}_0D_t^{-\gamma}p(x,t)$ is $q(x,s)/s^\gamma$. The above equation can be rewritten as

$$\frac{\partial^2}{\partial x^2}q(x,s) + A\frac{\partial}{\partial x}q(x,s) + Bq(x,s) = -\frac{s^{\gamma-1}}{K}\delta(x-a), \tag{15}$$

where

$$A = -\frac{\mu s^{\gamma-1}}{K}; \quad B = -\frac{s^\gamma}{K}. \tag{16}$$

Since $K > 0$, we have

$$\lambda^2 \equiv A^2 - 4B = \frac{\mu^2 s^{2\gamma-2}}{K^2} + 4\frac{s^\gamma}{K} \geq 0.$$

Therefore two independent solutions of the homogeneous equation corresponding to (15) are given by [29]

$$q_1(x,s) = \exp[x(\lambda - A)/2]; \quad q_2(x,s) = \exp[x(-\lambda - A)/2].$$

Consequently, the general solution of (15) satisfying all the boundary and initial conditions [cf. (4)] is given by

$$q(x,s) = \frac{s^{\gamma-1}}{K\lambda}e^{-A(x-a)/2}[e^{-\lambda|x-a|/2} - e^{-\lambda(x+a)/2}]. \tag{17}$$

To obtain the Laplace transform of the FPT density function, we take the Laplace transform of (5) to get

$$\phi_{\gamma,\mu}(s) = -s\int_0^\infty dx\, q(x,s) + \int_0^\infty dx\, p(x,0).$$

Here we have used the fact that Laplace transform of $dp(x,t)/dt$ is given by [30] $sq(x,s) - p(x,0)$. Since $p(x,0) = \delta(x-a)$ [cf. (4)], we obtain

$$\phi_{\gamma,\mu}(s) = 1 - s\int_0^\infty dx\, q(x,s).$$

Substituting for $q(x,s)$ from (17), we get

$$\begin{aligned}\phi_{\gamma,\mu}(s) = 1 - \frac{s^\gamma}{K\lambda}\Bigg[&\int_0^a dx\, e^{-A(x-a)/2}e^{-\lambda(a-x)/2} \\ &+ \int_a^\infty dx\, e^{-A(x-a)/2}e^{-\lambda(x-a)/2}\Bigg] \\ +\frac{s^\gamma}{K\lambda}&\int_0^\infty dx\, e^{-A(x-a)/2}e^{-\lambda(x+a)/2}.\end{aligned}$$

The integrals can be easily evaluated to finally give [upon using (16)]

$$\phi_{\gamma,\mu}(s) = \exp\left[-\frac{a\mu s^{\gamma-1}}{2K} - a\sqrt{\frac{\mu^2 s^{2\gamma-2}}{4K^2} + \frac{s^\gamma}{K}}\right].$$

This completes the proof of the theorem.

For $0 < \gamma < 1$, $\phi_{\gamma,\mu}(s)$ is not a completely monotone function [14] and hence is not a Laplace transform of a probability density function. For $1 \leq \gamma < 2$, we have not been able to prove rigorously that $\phi_{\gamma,\mu}(s)$ is completely monotone. However, we have calculated the first hundred derivatives of $\phi_{\gamma,\mu}(s)$ using the symbolic manipulation program Mathematica and find that all of them satisfy the monotonicity condition. Hence we conjecture that $\phi_{\gamma,\mu}(s)$ is the Laplace transform of a probability density function for $1 \leq \gamma < 2$. Henceforth, we restrict ourselves to this parameter range.

Corollary 1. *The mean first passage time* $\mathbf{E}(T)$ *for Levy type anomalous diffusion with* $1 \leq \gamma < 2$ *and drift* $|\mu|$ *towards the barrier is given by*

$$\mathbf{E}(T) = \frac{a}{|\mu|}.$$

The variance is infinite for $\gamma > 1$ *whereas for* $\gamma = 1$ *(ordinary diffusion) it is given by the standard result*

$$\mathrm{Var}(T) = \frac{\sigma^2 a}{|\mu|^3}.$$

For drift away from the barrier, mean and variance are infinite for $1 \leq \gamma < 2$.

Proof. First consider the case where the drift is towards the barrier. This implies that $\mu < 0$ since in our formulation the diffusive process starts at $x = a > 0$ and the barrier is at $x = 0$. In this case, the FPT density function can be written as [cf. (14)]

$$\phi_{\gamma,\mu}(s) = e^{ag(s)}, \tag{18}$$

where

$$g(s) = \frac{|\mu| s^{\gamma-1}}{2K} - \frac{|\mu| s^{\gamma-1}}{2K}\sqrt{1 + \frac{4Ks^{2-\gamma}}{\mu^2}}. \tag{19}$$

The mean first passage time is given by

$$\mathbf{E}(T) = -\frac{d\phi_{\gamma,\mu}(s)}{ds}\Big|_{s=0}. \tag{20}$$

From Eqs. (18) and (19), we have

$$\frac{d\phi_{\gamma,\mu}(s)}{ds} = \left[\frac{|\mu|(\gamma-1)s^{\gamma-2}a}{2K}\left(1 - \sqrt{1 + \frac{4Ks^{2-\gamma}}{\mu^2}}\right) - \frac{(2-\gamma)a(1 + \frac{4Ks^{2-\gamma}}{\mu^2})^{-1/2}}{|\mu|}\right] e^{ag(s)}.$$

We need to find the limiting value of the above expression as $s \to 0$. First consider $e^{ag(s)}$. As $s \to 0$, we can expand the square root in Eq. (19) to give

$$g(s) = -\frac{s}{|\mu|} + \frac{Ks^{3-\gamma}}{|\mu|^3} - \cdots$$

Hence $g(s) \to 0$ as $s \to 0$. Consequently, $e^{ag(s)} \to 1$ as $s \to 0$. Performing similar expansions for the other terms in (21), we finally obtain [cf. (20)]

$$\mathbf{E}(T) = \frac{a}{|\mu|}.$$

Note that the mean first passage time is independent of γ.

The variance is obtained as follows:

$$\mathrm{Var}(T) = \mathbf{E}(T^2) - [\mathbf{E}(T)]^2. \tag{21}$$

Therefore we need to evaluate $\mathbf{E}(T^2)$. This is given by

$$\mathbf{E}(T^2) = \frac{d^2\phi_{\gamma,\mu}(s)}{ds^2}\bigg|_{s=0}. \tag{22}$$

Now

$$\frac{d^2\phi_{\gamma,\mu}(s)}{ds^2} = \left[a\frac{d^2g(s)}{ds^2} + \left(a\frac{dg(s)}{ds}\right)^2\right] e^{ag(s)}. \tag{23}$$

Consider the first term. The second derivative of $g(s)$ is given by

$$\begin{aligned}\frac{d^2g(s)}{ds^2} &= \frac{|\mu|(\gamma-1)(\gamma-2)s^{\gamma-3}}{2K}\left(1-\sqrt{1+\frac{4Ks^{2-\gamma}}{\mu^2}}\right)\\ &\quad - \frac{(\gamma-1)(2-\gamma)}{|\mu|s}\left(1+\frac{4Ks^{2-\gamma}}{\mu^2}\right)^{-1/2}\\ &\quad + \frac{2K(2-\gamma)^2s^{1-\gamma}}{|\mu|^3}\left(1+\frac{4Ks^{2-\gamma}}{\mu^2}\right)^{-3/2}.\end{aligned}$$

Expanding all terms we get

$$\begin{aligned}\frac{d^2g(s)}{ds^2} &= -\frac{|\mu|(\gamma-1)(\gamma-2)s^{\gamma-3}}{2K}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdots(2n-3)}{2^n n!}\left(\frac{4Ks^{2-\gamma}}{\mu^2}\right)^n\\ &\quad - \frac{(\gamma-1)(2-\gamma)}{|\mu|s}\sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!}\left(\frac{4Ks^{2-\gamma}}{\mu^2}\right)^n\\ &\quad + \frac{2K(2-\gamma)^2s^{1-\gamma}}{|\mu|^3}\sum_{n=0}^{\infty}\frac{(-1)^n 1\cdot 3\cdots(2n+1)}{2^n n!}\left(\frac{4Ks^{2-\gamma}}{\mu^2}\right)^n.\end{aligned}$$

After considerable manipulation, this can be rewritten as

$$\begin{aligned}\frac{d^2g(s)}{ds^2} &= \frac{2K(2-\gamma)s^{1-\gamma}}{|\mu|^3}\sum_{n=0}^{\infty}\left[\frac{n(2-\gamma)+(3-\gamma)}{(n+2)}\right]\\ &\quad \frac{(-1)^n 1\cdot 3\cdots(2n+1)}{2^n n!}\left(\frac{4Ks^{2-\gamma}}{\mu^2}\right)^n.\end{aligned}$$

Thus the first term in (23) is given by

$$a\frac{d^2g(s)}{ds^2}e^{ag(s)} = \frac{2Ka(2-\gamma)s^{1-\gamma}}{|\mu|^3}e^{ag(s)}\sum_{n=0}^{\infty}\left[\frac{n(2-\gamma)+(3-\gamma)}{(n+2)}\right] \frac{(-1)^n 1\cdot 3\cdots(2n+1)}{2^n n!}\left(\frac{4Ks^{2-\gamma}}{\mu^2}\right)^n. \tag{24}$$

For $\gamma = 1$, the case of ordinary diffusion, we obtain from the above equation

$$a\frac{d^2g(s)}{ds^2}e^{ag(s)}\bigg|_{s=0} = \frac{\sigma^2 a}{|\mu|^3}.$$

Here we have also used the fact that $K = \sigma^2/2$ for $\gamma = 1$. On the other hand, for $1 < \gamma < 2$, as $s \to 0$ the prefactor multiplying the sum in (24) diverges whereas the sum itself is finite and bounded away from zero. Consequently, for $1 < \gamma < 2$ the first term in (23) diverges as $s \to 0$.

Next, we consider the second term in (23). We already know the limiting behaviour of this term for $1 \le \gamma < 2$ as $s \to 0$ from our earlier analysis for mean first passage time, namely,

$$\left(\frac{dg(s)}{ds}\right)^2 e^{ag(s)}\bigg|_{s=0} = \frac{a^2}{\mu^2}.$$

Substituting the above results in (22), for $\gamma = 1$ we obtain

$$\mathbf{E}(T^2) = \frac{\sigma^2 a}{|\mu|^3} + \frac{a^2}{\mu^2}.$$

Therefore [cf. (21)]

$$\mathrm{Var}(T) = \sigma^2 a/|\mu|^3,$$

for $\gamma = 1$. For $1 < \gamma < 2$, the first term in (23) diverges as $s \to 0$ whereas the second term is finite. Therefore $\mathbf{E}(T^2)$ diverges. Hence the variance also diverges.

Finally, consider the case where the drift is away from the barrier. This implies that $\mu > 0$ in our formulation. Now the FPT density function can be written as in (18) where

$$g(s) = -\frac{|\mu|s^{\gamma-1}}{2K} - \frac{|\mu|s^{\gamma-1}}{2K}\sqrt{1+\frac{4Ks^{2-\gamma}}{\mu^2}}.$$

Performing the same analysis as above, it is easily seen that the mean and variance diverge for $1 \le \gamma < 2$. This completes the proof.

Corollary 2. *For ordinary diffusion ($\gamma = 1$) with drift μ and diffusion constant K, the first passage time density function is given by*

$$f_{1,\mu}(t) = \frac{a}{\sqrt{4\pi K t^3}}\exp\left[-\frac{(a+\mu t)^2}{4Kt}\right], \quad a > 0, \quad t > 0.$$

Proof. For $\gamma = 1$, the Laplace transform of the first passage time density function reduces to [cf. Eq. (14)]

$$\phi_{1,\mu}(s) = \exp\left[-\frac{a}{2K}(-\mu - \sqrt{\mu^2 + 4Ks}\right]. \tag{25}$$

Now the inverse Laplace transform of $\exp(-\alpha\sqrt{s}),\ \alpha \geq 0$ is [30]

$$\frac{\alpha}{2\sqrt{\pi t^3}} \exp(-\alpha^2/4t).$$

Using this result, we can easily perform the inverse Laplace transform of (25) to obtain

$$f_{1,\mu}(t) = \frac{a}{\sqrt{4\pi K t^3}} \exp\left[-\frac{(a+\mu t)^2}{4Kt}\right], \quad a > 0, \quad t > 0.$$

We comment that, if the starting point of the diffusion is chosen at $x(0) = 0$, a negative μ will be used in the above equation and we recover the expected inverse Gaussian density, (1), for the FPT density function of a ordinary diffusion with drift. This completes the proof.

5 Summary

We have derived the explicit first passage time density function in terms of H-functions using a fractional Fokker-Planck equation formalism for zero drift Levy type anomalous diffusion. This was shown to be a proper probability density. For Levy type anomalous diffusion with drift and $1 \leq \gamma < 2$, the moment generating function (Laplace transform of the first passage time density function) was obtained. The mean first passage time was derived and shown to be independent of γ. The FPT density function for ordinary diffusion with drift was derived as a special case.

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Appendix: Properties of H-functions

The H-function has the following remarkable properties [21] which we will use later.

Property 1. The H-function is symmetric in the pairs $(a_1, A_1), \ldots, (a_n, A_n)$, likewise $(a_{n+1}, A_{n+1}), \ldots, (a_p, A_p)$; in $(b_1, B_1), \ldots, (b_m, B_m)$ and in (b_{m+1}, B_{m+1}), $\ldots, (b_q, B_q)$.

Property 2. Provided $n \geq 1$ and $q > m$,

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_1,A_1), (a_2,A_2), & \cdots, & (a_p,A_p)\\ (b_1,B_1), \quad \cdots, & (b_{q-1},B_{q-1}), & (a_1,A_1)\end{matrix}\right.\right)$$
$$= H_{p-1,q-1}^{m,n-1}\left(z\left|\begin{matrix}(a_2,A_2), \cdots, & (a_p,A_p)\\ (b_1,B_1), \cdots, & (b_{q-1},B_{q-1})\end{matrix}\right.\right).$$

Property 3. Provided $m \geq 2$ and $p > n$,

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_1,A_1), \quad \cdots, & (a_{p-1},A_{p-1}), & (b_1,B_1)\\ (b_1,B_1), (b_2,B_2), & \cdots, & (b_q,B_q)\end{matrix}\right.\right)$$
$$= H_{p-1,q-1}^{m-1,n}\left(z\left|\begin{matrix}(a_1,A_1), \cdots, & (a_{p-1},A_{p-1})\\ (b_2,B_2), \cdots, & (b_q,B_q)\end{matrix}\right.\right).$$

Property 4.

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_j,A_j)_{j=1,\ldots,p}\\ (b_j,B_j)_{j=1,\ldots,q}\end{matrix}\right.\right) = H_{q,p}^{n,m}\left(\frac{1}{z}\left|\begin{matrix}(1-b_j,B_j)_{j=1,\ldots,q}\\ (1-a_j,A_j)_{j=1,\ldots,p}\end{matrix}\right.\right).$$

Property 5. For $k > 0$,

$$\frac{1}{k}H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_j,A_j)_{j=1,\ldots,p}\\ (b_j,B_j)_{j=1,\ldots,q}\end{matrix}\right.\right) = H_{p,q}^{m,n}\left(z^k\left|\begin{matrix}(a_j,kA_j)_{j=1,\ldots,p}\\ (b_j,kB_j)_{j=1,\ldots,q}\end{matrix}\right.\right).$$

Property 6.

$$z^{\rho}H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_j,A_j)_{j=1,\ldots,p}\\ (b_j,B_j)_{j=1,\ldots,q}\end{matrix}\right.\right) = H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_j+\rho A_j,A_j)_{j=1,\ldots,p}\\ (b_j+\rho B_j,B_j)_{j=1,\ldots,q}\end{matrix}\right.\right).$$

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