GENERALIZED EMITTANCE INVARIANTS

Govindan Rangarajan, Filippo Neri, and Alex Dragt
Department of Physics and Astronomy, University of Maryland
College Park, MD 20742

ABSTRACT

The mean square emittance is useful for analyzing the behavior of beams described by a two-dimensional phase space (or of beams described by a higher dimensional phase space but for which the various degrees of freedom are uncoupled) because it remains invariant under beam transport through any optical system for which there are no nonlinear forces. Using Lie algebraic properties of the symplectic group, we show that in the general case of a six-dimensional phase space (including possible coupling between all degrees of freedom) the concept of mean square emittance can be generalized to produce three invariants. These invariants (which can be viewed as eigen-emittances) are made out of second order moments, and can be shown to form a complete set. They should be very useful in the analysis of general beam transport. Finally, the Lie algebraic methods can be extended to make infinitely many invariants out of cubic and higher order moments.

I. INTRODUCTION

The main purpose of this paper is to provide a Lie algebraic treatment of moments of particle distributions and invariants constructed out of these moments. In Section II, moments are defined and their evolution under beam transport is determined. In Section III, the concept of mean square emittance is generalized to obtain quantities that remain invariant under full six dimensional linear beam transport with couplings between the three degrees of freedom. Finally, a method to construct higher order invariants is given.

II. BASIC CONCEPTS

Let $z = (x, p_x, y, p_y, \tau, p_\tau)$ be the six dimensional vector describing the location of a particle in phase space. Consider the action of a linear beam transport system on this particle. Its effect can be described by a linear transfer map M. Denoting the initial and final locations of the particle by z^i and z^f respectively, we can write

$$z^f = Mz^i. \tag{2.1}$$

If our beam transport system is Hamiltonian, M is a 6×6 symplectic matrix, and satisfies the equation[1]

$$\tilde{M}JM = J \tag{2.2}$$

where $\tilde{M}$ is the transpose of M and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.3}$$

Here I is the 3×3 identity matrix.

In the following we derive the basic equation for transport of moments. Let $h(z)$ be the initial distribution function describing particle density in phase space with coordinates z. Also, let $P_\alpha(z)$, where α is some running index, denote a complete set of homogeneous polynomials in z. Then one can define a set of initial moments w^i_α by the rule

$$w^i_\alpha \equiv \int d^6z\, h(z) P_\alpha(z). \tag{2.4}$$

Now suppose the particle distribution $h(z)$ is transported through the system described by the linear transfer map M. Then the final distribution at the end of the system is given by $h(M^{-1}z)$. Correspondingly, the final moments are given by the expressions

$$w^f_\alpha = \int d^6z\, h(M^{-1}z) P_\alpha(z). \tag{2.5}$$

After some manipulation using the symplectic property of M, we get

$$w^f_\alpha = \int d^6z'\, h(z') P_\alpha(Mz'). \tag{2.6}$$

Also by completeness of $P_\alpha(z)$ one has a relation of the form

$$P_\alpha(Mz) = D(M)_{\alpha\beta}\, P_\beta(z). \tag{2.7}$$

Substituting this in Eq. (2.6) we finally get

$$w^f_\alpha = D(M)_{\alpha\beta}\, w^i_\beta. \tag{2.8}$$

This is the basic equation for moment transport.

III. KINEMATIC MOMENT INVARIANTS

A. DEFINITIONS

We are now in a position to define moment invariants. Rewrite Eq. (2.8) as

$$w^f = D(M)\, w^i. \tag{3.1}$$

Suppose a function $I(w)$ has the following property

$$I(D(M)w) = I(w) \tag{3.2}$$

for all M. Then I is called a ***kinematic moment invariant.***

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Another important concept is that of moment equivalence classes. Suppose there exists an M such that

$$w^b = D(M) w^a. \tag{3.3}$$

Then we write

$$w^b \sim w^a. \tag{3.4}$$

This relation is an equivalence relation. Let $[w]$ be the set of all w^b such that $w^b \sim w^a$. The set $[w]$ is called the equivalence class of w.

This leads us to the observation that a kinematic moment invariant is a class function i.e. $I(w^b) = I(w^a)$ if $w^b \sim w^a$. Thus

$$I = I([w]). \tag{3.5}$$

From the above discussion, we conclude that the number of functionally independent kinematic moment invariants is equal to the dimensionality of the set of equivalence classes.

B. QUADRATIC MOMENT INVARIANTS

An example of a kinematic moment invariant is the familiar two dimensional mean square emittance defined as

$$\epsilon^2 = < x^2 >< p_x^2 > -(< xp_x >)^2. \tag{3.6}$$

Using Eq. (2.8) it can be shown that mean square emittance remains invariant under two dimensional linear beam transport. One of our goals is to generalize this to obtain quantities that remain invariant under full six dimensional linear beam transport with couplings between the x, y, τ degrees of freedom.

For the present, let us restrict our attention to quadratic moments. Given any set of quadratic moments w, we can find[2] an equivalent set w^* (i.e. $w \sim w^*$) such that the moments w^* have the following special properties

$$< z_a z_b >^* = 0 \quad \text{if} \quad a \neq b, \tag{3.7}$$

$$< xx >^* = < p_x p_x >^*, \tag{3.8a}$$

$$< yy >^* = < p_y p_y >^*, \tag{3.8b}$$

$$< \tau\tau >^* = < p_\tau p_\tau >^*. \tag{3.8c}$$

This shows that there are three equivalence classes of quadratic moments and hence three functionally independent kinematic invariants made from quadratic moments.

We can take the three independent invariants to be the eigen mean square emittances ϵ_x^2, ϵ_y^2, and ϵ_τ^2 defined as follows:

$$\epsilon_x^2 = < xx >^* < p_x p_x >^*, \tag{3.9a}$$

$$\epsilon_y^2 = < yy >^* < p_y p_y >^*, \tag{3.9b}$$

$$\epsilon_\tau^2 = < \tau\tau >^* < p_\tau p_\tau >^*. \tag{3.9c}$$

Then any kinematic invariant made of quadratic moments can be written in the form

$$I_2 = I_2(\epsilon_x^2, \epsilon_y^2, \epsilon_\tau^2). \tag{3.10}$$

Another choice is to take the functions $I_2^{(2)}[w]$, $I_2^{(4)}[w]$, and $I_2^{(6)}[w]$ defined by the relations

$$I_2^{(2)}[w] = \epsilon_x^2 + \epsilon_y^2 + \epsilon_\tau^2, \tag{3.11a}$$

$$I_2^{(4)}[w] = \epsilon_x^4 + \epsilon_y^4 + \epsilon_\tau^4, \tag{3.11b}$$

$$I_2^{(6)}[w] = \epsilon_x^6 + \epsilon_y^6 + \epsilon_\tau^6. \tag{3.11c}$$

An alternative method for obtaining these invariants is outlined below. The advantage of this method is that it can be easily generalized to construct invariants made of higher order moments. Let

$$I_2^{(n)}[w] = \frac{1}{2}(-1)^{\frac{n}{2}} \mathrm{tr}(ZJ)^n \tag{3.12}$$

where Z is a 6×6 symmetric matrix whose elements are defined as

$$Z_{ab} = < z_a z_b > \tag{3.13}$$

and J is the fundamental symplectic matrix defined in Eq. (2.3). It can be shown that $I_2^{(n)}[w]$ is invariant under linear beam transport. Further, $I_2^{(n)}[w]$ for $n = 2$, 4 and 6 correspond to the three independent invariants listed in Eq. (3.11). In particular, one finds the result[3]

$$\begin{aligned} I_2^{(2)}[w] = & < x^2 >< p_x^2 > - < xp_x >^2 + < y^2 >< p_y^2 > \\ & - < yp_y >^2 + < \tau^2 >< p_\tau^2 > - < \tau p_\tau >^2 \\ & + 2 < xy >< p_x p_y > -2 < xp_y >< yp_x > \\ & + 2 < x\tau >< p_x p_\tau > -2 < xp_\tau >< \tau p_x > \\ & + 2 < y\tau >< p_y p_\tau > -2 < yp_\tau >< \tau p_y >. \end{aligned} \tag{3.14}$$

The expressions for $I_2^{(4)}$ and $I_2^{(6)}$ in six dimensional phase space are not listed since they are very long.

C. HIGHER ORDER MOMENT INVARIANTS

We now generalize the above concepts to construct invariants made of higher order moments. For simplicity, we deal only with invariants made of cubic and quartic moments. Generalizations to moments of arbitrary order can be found elsewhere[2].

Let

$$I_3^{(2n)}[w] = \mathrm{tr}[(Z^{(3)} J Z^{(3)} JJ)^n], \tag{3.15}$$

$$I_4^{(n)}[w] = \mathrm{tr}[(Z^{(4)} JJ)^n] \tag{3.16}$$

where $Z^{(3)}$ and $Z^{(4)}$ are third and fourth rank tensors whose elements are cubic and quartic moments respectively:

$$Z^{(3)} = < z_a z_b z_c >, \tag{3.17}$$

$$Z^{(4)} = < z_a z_b z_c z_d >. \tag{3.18}$$

It can be shown that I_3 and I_4 are kinematic moment invariants. In two dimensional phase space, we find the functionally independent cubic and quartic moment invariants to be as follows (with the leading coefficient normalized to be equal to +1):

$$
\begin{aligned}
I_3^{(4)}[w] =& < x^3 >^2 < p_x^3 >^2 - 3 < x^2 p_x >^2 < x p_x^2 >^2 \\
& + 4 < x^3 >< x p_x^2 >^3 + 4 < x^2 p_x >^3 < p_x^3 > \\
& - 6 < x^3 >< x^2 p_x >< x p_x^2 >< p_x^3 >, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
I_4^{(2)}[w] =& < x^4 >< p_x^4 > + 3 < x^2 p_x^2 >^2 \\
& - 4 < x^3 p_x >< x p_x^3 >, \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
I_4^{(3)}[w] =& < x^4 >< p_x^4 >< x^2 p_x^2 > - < x^4 >< x p_x^3 >^2 \\
& - < x^2 p_x^2 >^3 - < x^3 p_x >^2 < p_x^4 > \\
& + 2 < x^3 p_x >< x p_x^3 >< x^2 p_x^2 > . \qquad (3.21)
\end{aligned}
$$

Invariants can also be constructed using moments of different orders. A simple example of such a mixed invariant combining linear and quadratic moments is given below:

$$
\begin{aligned}
I_{1,2}^{(2,1)}[w] =& < x^2 >< p_x >^2 - 2 < x p_x >< x >< p_x > \\
& + < p_x^2 >< x >^2 . \qquad (3.22)
\end{aligned}
$$

Such mixed moment invariants become important when the beam transport system contains misaligned optical elements and other "zeroth" order effects. When a particle distribution is transported through such a system, none of the $I_m^{(n)}[w]$'s given above remain invariant. Instead it is combinations like $I_2^{(2)}[w] + I_{1,2}^{(2,1)}[w]$ that remain invariant. Such combinations always involve a mixed "invariant".

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