Algebraic topology midterm solutions

1

1.a

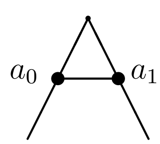
true. Any closed surface is represented by a planar polygon with some side identifications. Since $S \setminus \{p\}$ is obtained by deleting a point from the interior of the polygon, it deform retracts to the boundary of the polygon which is a wedge sum of circles after the identifications. Thus $S \setminus \{p\}$ is homotopy equivalent to a wedge sum of circles, and hence $\pi 1_(S \setminus \{p\})$ is a free group.

1.b

false.take $X = \mathbb{R}^2$.

1.c

true. A and P are both homotopy equivalent to circle (but a deform-retraction that collapses their “legs", so they are also homotopy equivalent.



now if you take out a_0 and a_1 points from A there are four connected components. But if you take out any two points from P, P will have atmost three connected components. Hence A and P are not homeomorphic.

1.d

true.

1.e

false. Any finite group, is not subgroup of any free group, since a free group does not have any element of finite order.(Later we shall see that any subgroup of free group is also free)

2

2.a

First prove that $\langle S \rangle$ is convex (easy).
Secondly if A is any convex set containing S ,$\langle S \rangle \subseteq A$.

Proof:let $U \in \langle S \rangle$ then

$$U = \sum_{k=1}^{m} t_k s_k \quad \sum t_k = 1 \quad 0 \leq t_k \leq$$

If $m = 1$, then $U \in A$, and $\langle S \rangle \subseteq A$.
If $m = 2$, then $U \in A$, because A is convex.
Let the result hold for $2 \leq k \leq m$,and

$$U = \sum_{k=1}^{m} t - k s_k$$

. Now $t_m = 1 - \sum_{j=1}^{m-1} t_j$. Hence

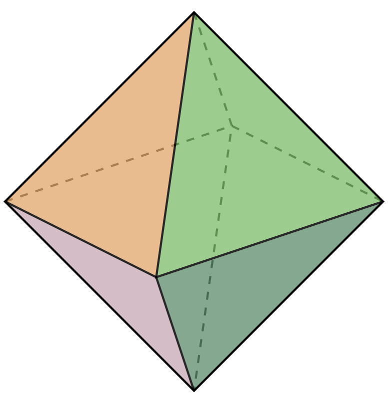
$$U = \sum_{j=1}^{m-1} t_k s_k + t_m s_m$$

$$W = \sum_{j=1}^{m-1} \frac{t_k}{1 -_m} s_k$$

$$\sum_{k=1}^{m-1} \frac{t_k}{1 - t_m} = 1 \implies W \in A$$

. Now $U = (1 - t_m)W + t_m s_k \in A$, by convexity.

2.b



First observe that:

1. v_i, u_j, w_k for $1 \leq i, j, k \leq 2$ are on the surface of S^2.
2. $(0, 0, 0)$ lies in the interior of the simplex.

Let l_x be the line joining the origin and x on the boundary of simplex. If we extend l_x in the direction of x it will intersect S^2 in exactly one point. Now let ϕ be defined as follows:

$$\phi : |K| \to S^2$$
$$\phi(x) = \frac{x}{\parallel x \parallel}$$

Observe that ϕ is continuous, injective and onto. Moreover S^2 is Hausdorff and boundary of the simplex is compact. Also, bijective continuous map from a compact set to a Hausdorff space is a homeomorphism.

Otherwise, prove that $|K|$ is a closed surface and see that $\chi(K) = 2$ (From classification of surfaces one can conclude that it is a sphere).

3

3.a

If $S^1 \times \partial D^2$ is a retract of X, then the inclusion $i : S^1 \times \partial D^2 \to X$ induces an injective homomorphism:

$$i_* : \pi_1(S^1 \times \partial D^2) \to \pi_1(X).$$

But $\pi_1(S^1 \times \partial D^2) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(X) = \mathbb{Z}$ and there is no injective homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Hence $S^1 \times \partial D^2$ is not a retract of X.

3.b

Consider the following map:

$$H : I \times (S^1 \times D^2) \to S^1 \times D^2$$
$$H(t, u, v) = (u, tv + (1-t)x_0)$$

where $u \in S^1, v \in D^2$. Then H is continuous and

$$H(1, u, v) = (u, v) = I_{S^1 \times D^2}$$
$$H(0, u, v) = (u, x_0) \in S^1 \times \{x_0\}$$

3.c

No. The subspace $A = \{x_1\} \times \partial D^2$ is not a retract of $X = S^1 \times D^2$. If there is a retraction $r : S^1 \times D^2 \to \{x_1\} \times \partial D^2$, then $r \circ i \equiv Id_A$ which implies that $r_* \circ i_* = Id_{\pi_1(A)}$, and in particular, i_* is injective. This is not possible as the image under i of the loop generating $\pi_1(A)$ in X is null-homotopic.

4

4.a

Let α be the generator of $\pi_1(M)$. Note that the boundary of the Möbius strip is α^2.

Using van Kampen theorem, the fundamental group is

$$\pi_1(X) = \langle a, b, \alpha | b^2 = 1, ab = 1, \alpha^2 = b \rangle$$

Using the relations to eliminate generators, we obtain:

$$\pi_1(X) = \langle \alpha | \alpha^4 = 1 \rangle \approx \mathbb{Z}_4$$

4.b

Take a square and identify all sides as shown in the next page.

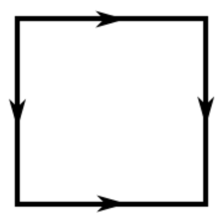


Figure 1: