S in $\mathbb{R}^n$ is called

Exercise Set - $\overline{V}$

1. Calculate all first order partial derivatives and the directional derivative $f'(\underline{x}, \underline{u})$ for each of the real-valued functions defined on $\mathbb{R}^n$ as follows:

   a) $f(\underline{x}) = \underline{a} \cdot \underline{x}$ where $\underline{a} \in \mathbb{R}^n$ is fixed,

   b) $f(\underline{x}) = \|\underline{x}\|^4$,

   c) $f(\underline{x}) = \underline{x} \cdot L(\underline{x})$ where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation,

   d) $f(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$.

2. Let f and g be real valued functions such that the directional derivatives $f'(\underline{c}, \underline{u})$ and $g'(\underline{c}, \underline{u})$ exist. Prove that the sum $f+g$ and the product fg have directional derivatives given by

$$(f+g)'(\underline{c}, \underline{u}) = f'(\underline{c}, \underline{u}) + g'(\underline{c}, \underline{u})$$

$$(fg)'(\underline{c}, \underline{u}) = f(\underline{c})\, g'(\underline{c}, \underline{u}) + g(\underline{c})\, f'(\underline{c}, \underline{u}).$$

$\mathbb{R}^n$ is called

3. Given n real-valued functions $f_1, \ldots, f_n$ each differentiable on an open interval (a, b) in $\mathbb{R}$, define f on

6 $$S = \{ (x_1, \ldots, x_n) : a < x_k < b, \ k = 1, \ldots, n \}$$

by $f(\underline{x}) = f_1(x_1) + \cdots + f_n(x_n)$. Prove that f is differentiable on S. Find a formula for $f'(\underline{x})(\underline{u})$ in terms of $f_1'(x_1), \ldots, f_n'(x_n)$ and $u_1, \ldots, u_n$.

4. Let f be a real-valued function, differentiable at a point $\underline{c}$ in $\mathbb{R}^n$ and assume that $\|\nabla f(\underline{c})\| \neq 0$. Prove that there is a unique unit vector $\underline{u}$ in $\mathbb{R}^n$ such that $|f'(\underline{c}, \underline{u})| = \|\nabla f(\underline{c})\|$ and that this is the unit vector for which $|f'(\underline{c}, \underline{u})|$ has its maximum value.

5. Compute the gradient vector ~~at~~ $\nabla f(x, y)$ at those points $(x, y) \in \mathbb{R}^2$ where it exists:

a) $$f(x,y) = \begin{cases} x^2 y^2 \log(x^2 + y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

b) $f(x,y) = \begin{cases} xy \sin \dfrac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

6. Let f and g be real-valued functions defined on $\mathbb{R}$ with continuous second derivatives f'' and g''. Define

$$F(x,y) = f(x + g(y)), \quad (x,y) \in \mathbb{R}^2.$$

Find formulas for all partials of F of first and second order in terms of the derivatives of f and g.

Verify that $(D_1 F)(D_{1,2} F) = (D_2 F)(D_{1,1} F)$.

7. If f and g have gradient vectors $\nabla f(\underline{x})$ and $\nabla g(\underline{x})$ at a point $\underline{x} \in \mathbb{R}^n$, show that the product function h defined by $h(\underline{x}) = f(\underline{x})\, g(\underline{x})$ also has a gradient vector at $\underline{x}$ and $\nabla h(\underline{x}) = f(\underline{x}) \nabla g(\underline{x}) + g(\underline{x}) \nabla f(\underline{x})$.

State and prove a similar result for f/g.

8. Let f be a differentiable function on $\mathbb{R}$ and let g be defined on $\mathbb{R}^3$ by $g(x,y,z) = x^2 + y^2 + z^2$. If h denotes the composite function $h = f \circ g$, show that

$$\| \nabla h(x,y,z) \|^2 = 4\, g(x,y,z) \left(f'(g(x,y,z)) \right)^2.$$

9. A function f defined on an open set S in $\mathbb{R}^n$ is called homogeneous of degree p over S if $f(\lambda \underline{x}) = \lambda^p f(\underline{x})$ for every real λ and every $\underline{x} \in S$ for which $\lambda \underline{x} \in S$. If such a function f is differentiable at $\underline{x}$, show that

$$\underline{x} \cdot \nabla f(\underline{x}) = p f(\underline{x}).$$

Also, prove the converse, i.e., show that if $\underline{x} \cdot \nabla f(\underline{x}) = p f(\underline{x})$ for all $\underline{x}$ in an open set S, then f ~~must~~ must be homogeneous of degree p over S.

10. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$ and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

11. Let $w = f(u, v)$ satisfy the Laplace equation $f_{uu} + f_{vv} = 0$. If $u = \frac{x^2 - y^2}{2}$ and $v = xy$, show that w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.