UM 102 (2025): Problem set 2

1. Given two subspaces W_1 and W_2 of a vector space V, define

$$W_1 + W_2 = \{w_1 + w_2 : w_i \in W_i, i = 1, 2\}.$$

   Prove or give a counterexample: if U_1, U_2 and W are subspaces of V and $U_1 + W = U_2 + W$, then $U_1 = U_2$.

2. Let $\mathcal{P}_m$ be the vector space of polynomials of degree up to m. Prove or disprove: there is a basis $\{p_o, p_1, p_2, p_3\}$ of $\mathcal{P}_3$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

3. Suppose that $p_0, p_1, \ldots, p_m$ are polynomials $\mathcal{P}_m$ such that $p_j(2) = 0 \ \forall j$. Show that $\{p_0, p_1, \ldots, p_m\}$ is not linearly independent.

4. If V is a finite dimensional vector space and $U_1, \ldots, U_m$ are subspaces of V, then

$$\dim U_1 + \cdots + U_m \leq \dim U_1 + \cdots + \dim U_m.$$

5. Suppose T is a linear transformation from an inner product space V to $\mathbb{C}$. If $u \in V$ is not in the kernel of T, prove that $V = \text{Kernel } T + \{cu : c \in \mathbb{C}\}$.

6. If T is a linear transformation from $\mathbb{C}^4$ to $\mathbb{C}^2$ such that

$$\text{Kernel } T = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1 = 5z_2 \text{ and } z_3 = 7z_4\},$$

   then show that T is onto.

7. Let $T : V \to V$ be a linear map such that both Kernel T and range T are finite-dimensional. Prove that V is finite-dimensional.

8. Find a polynomial $q \in P_2(\mathbb{R})$ such that $\int_0^1 p(x) \cos \pi x dx = \int_0^1 p(x)q(x)dx$ for all $p \in P_2(\mathbb{R})$.

9. If V is a finite-dimensional vector space and $\dim V > 1$, prove that the set of non-invertible linear transformations is not a subspace of the vector space of all linear transformations on V.