PROBLEM SET 4 FOR UM 102/UMA102

MAINLY ON EIGENVALUES.

(1) Let T be the $n \times n$ matrix whose all entries are 1. Find all the eigenvalues and eigenvectors of T

(2) Prove that an $n \times n$ matrix of rank $k < n$ has at most $k+1$ distinct eigenvalues.

(3) Prove that the eigenvalues of the inverse of an invertible matrix are the reciprocals of the eigenvalues of the original matrix.

(4) Let $\sigma(A)$ denote the set of eigenvalues of A. If p is a polynomial and T is an $n \times n$ matrix, show that $\sigma(p(A)) = p(\sigma(A))$ which by definition is the set of all $p(\lambda)$ such that λ is in $\sigma(A)$.

(5) Prove that ST and TS have the same eigenvalues for any two square matrices S and T.

(6) Let $n \geq 3$, let B be an $(n-2) \times (n-2)$ matrix and let λ, μ be complex numbers. Consider the block matrix

$$A = \begin{pmatrix} \lambda & \star & \star \\ 0 & \mu & 0 \\ 0 & \star & B \end{pmatrix}.$$

Express the characteristic polynomial of A in terms of the characteristic polynomial of B.

(7) Let $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Show that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$.

(8) If A is an idempotent matrix, i.e., $A^2 = A$, show that each eigenvalue of A is either 0 or 1. Explain why I is the only invertible idempotent matrix.

(9) Suppose that λ is an eigenvalue of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if either column of $\begin{pmatrix} d-\lambda & -b \\ -c & a-\lambda \end{pmatrix}$ is non-zero, then it is an eigenvector of A associated with λ.

(10) Prove or give a counterexample: if W is a subspace of V that is invariant under every linear transformation on V, then $W = \{0\}$ or $W = V$.