ERRATUM TO 'NON-UNIFORMLY FLAT AFFINE ALGEBRAIC HYPERSURFACES'

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ABSTRACT. Correcting an erroneous result in [PV-2021], we prove that the affine algebraic hypersurfaces $\{xy^2 = 1\} \subset \mathbb{C}^2$ and $\{z = xy^2\} \subset \mathbb{C}^3$ are not interpolating for the Gaussian weight.

Let (X, g) be a Hermitian manifold, $(L, e^{-\varphi}) \to X$ a Hermitian holomorphic line bundle, and $Z \subset X$ a a complex analytic subvariety $Z \subset X$. One says that Z is an *interpolation subvariety*, or simply *interpolating*, for the above data if the restriction map

$$\mathscr{R}_Z : H^0(X, \mathcal{O}_X(L)) \to H^0(Z, \mathcal{O}_Z(L))$$

induces a surjective map of the Bergman spaces

$$\mathscr{R}_Z : \mathscr{B}_n(X, \varphi) \to \mathfrak{B}_d(Z, \varphi)$$

(see [PV-2021] for the notation and more details). In the present note we consider only the case $X = \mathbb{C}^2$ or $X = \mathbb{C}^3$ with the Euclidean metric ω_o. Since in this case any line bundle is trivial, metrics have a well-defined logarithm, and we call the function $\varphi := -\log e^{-\varphi}$ a *weight function*.

In [PV-2021, Theorems 2 and 3] the second and third authors (Pingali and Varolin) claimed that for any smooth weight function φ satisfying $0 < m\omega_o \leq \sqrt{-1}\partial\bar{\partial}\varphi \leq M\omega_0$ the (non-uniformly flat) manifolds

$$C_2 = \{(x, y) \in \mathbb{C}^2 \mid xy^2 = 1\} \subset \mathbb{C}^2 \quad \text{and} \quad S = \{(x, y, z) \in \mathbb{C}^3 \mid z = xy^2\} \subset \mathbb{C}^3$$

are interpolating. The proof of the claim rests heavily on Lemma 3.2 which aims to generalize the QuimBo trick [BOC-1995]. Unfortunately, Lemma 3.2 is false. (However, for Theorems 1 and 4 we do not need Lemma 3.2. Instead, [L-1997, Lemma 6] in conjunction with elliptic regularity is enough.) In fact, we prove that the negations of Theorem 2 and Theorem 3 in [PV-2021] are true.

Theorem 1.1. *The curve $C_2 \subset \mathbb{C}^2$ is not interpolating with respect to the Gaussian weight $|\cdot|^2$.*

An application of [PV-2021, Theorem 6.1] establishes the following result.

Corollary 1.2. *The surface $S \subset \mathbb{C}^3$ is not interpolating with respect to the Gaussian weight $|\cdot|^2$.*

Proof of Theorem 1.1. Let $f_n \in \mathcal{O}(C_2)$ be defined by $f_n(x, y) = y^{-(2n+1)}$. Then

$$(1) \qquad ||f_n||^2 = \int_{\mathbb{C}^*} \frac{e^{-(|y|^{-4}+|y|^2)}}{|y^{2n+1}|^2} \left(1 + \frac{4}{|y|^6}\right) dA(y) = \pi \int_{r=0}^{\infty} \frac{e^{-(r+r^{-2})}}{r^{2n+1}} \left(1 + \frac{4}{r^3}\right) dr.$$

For positive numbers s and t, integration-by-parts shows that

$$(2) \qquad \int_0^{\infty} e^{-(sr+tr^{-2})} \left(1 + \frac{4}{r^3}\right) dr = \left(1 + \frac{2s}{t}\right) \int_0^{\infty} e^{-(sr+tr^{-2})} dr.$$

Applying $(\frac{\partial}{\partial t})^{n+1}\frac{\partial}{\partial s}$ to (2) and then setting $s=t=1$ yields

$$\int_0^\infty r^{-(2n+1)}e^{-(r+r^{-2})}\left(1+4r^{-3}\right)dr$$

$$=\int_0^\infty r^{-2n-1}e^{-(r+r^{-2})}dr+2(n+1)!\int_0^\infty (r-1)e^{-(r+r^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2k}}{k!}dr. \tag{3}$$

Now, for $r>0$, $r^{-(2n+2)}e^{-r^{-2}}\leq (n+1)^{n+1}e^{-(n+1)}\sim\frac{(n+1)!}{\sqrt{2\pi(n+1)}}$ by Stirling's Formula, so

$$\int_0^\infty r^{-2n-1}e^{-(r+r^{-2})}dr\leq\frac{2\pi(n+1)!}{\sqrt{(n+1)}}\int_0^\infty re^{-r}dr=\frac{2\pi(n+1)!}{\sqrt{(n+1)}}$$

for large enough n. Together with (1) and (3), one therefore has

$$||f_n||^2\leq 2\pi(n+1)!\left(\frac{1}{\sqrt{n+1}}+\int_0^\infty (r-1)e^{-(r+r^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2k}}{k!}dr\right)<\infty. \tag{4}$$

To achieve our contradiction, suppose C_2 is interpolating. Then there exists $F_n\in\mathscr{B}_2$ such that

$$F_n|_{C_2}=f_n \quad\text{and}\quad ||F_n||\leq C||f_n|| \tag{5}$$

for some $C>0$ independent of n. Writing $F_n(x,y)=\sum_{i,j\geq 0}c_{ij}x^iy^j$, we have

$$y^{-(2n+1)}=\sum_{i,j\geq 0}c_{ij}y^{-2i}y^j=\sum_{i,j\geq 0}c_{ij}y^{-(2i-j)}=\sum_{2i-j=2n+1}c_{ij}y^{-(2i-j)}. \tag{6}$$

Setting $y=1$ shows that $\sum_{k\geq 1}c_{k+n,2k-1}=1$, and hence $|c_{m+n,2m-1}|\geq 2^{-(m+1)}$ for some $m\in\mathbb{N}$. Therefore

$$||F_n||^2\geq |c_{m+n,2m-1}|^2(m+n)!(2m-1)!\geq\frac{(n+1)!}{2^4}. \tag{7}$$

From (4), (5) and (7) we conclude that for $n>>0$

$$2^{-4}\leq C\left(\frac{1}{\sqrt{n+1}}+\int_0^\infty (r-1)e^{-(r+r^{-2})}\sum_{k=0}^{n+1}\frac{r^{-2k}}{k!}dr\right)=O\left(\frac{1}{\sqrt{n+1}}\right).$$

This is the desired contradiction. □

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