LECTURES 5 AND 6

1. Lecture 5 (First Chern form, Kähler connection, and curvature)

Suppose $\tilde{h}$ is another Hermitian metric. Note that $\frac{\tilde{h}_\alpha}{h_\alpha} = \frac{\tilde{h}_\beta}{h_\beta}$. Hence $\tilde{h} = he^{-\phi}$ where ϕ is some smooth globally defined function. Note that $c_1(\tilde{h}) = c_1(h) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$. In other words, $[c_1(h)]$ (the De Rham cohomology class) is independent of the metric chosen !! This topological quantity is called the first Chern class of the line bundle L.

It seems that the first Chern form almost gives us a Kähler form except for the point that it may not be positive-definite. We define the following : A holomorphic line bundle is said to positive is $c_1(L)(h)$ is positive for some Hermitian metric h (likewise, negative). A $(1,1)$ De Rham cohomology class $[\omega]$ is said to be positive if there is a Kähler form in it, i.e., there is a Kähler form ω' such that $\omega' = \omega + d\eta$.

Exercise : Using the $\partial\bar{\partial}$ lemma, show that if $[\omega] = [c_1(L)]$ then there is a smooth Hermitian metric h so that $c_1(h) = \omega$, i.e., every form in the first Chern class can be realised using a Hermitian metric.

By the way,

Theorem 1.1 (Kodaira's embedding theorem)**.** *A compact complex manifold can be holomorphically embedded as a submanifold in $\mathbb{CP}^n$ iff it has a positive holomorphic line bundle L.*

The Fubini-Study metric (up to a factor) as we defined it is the first Chern form of $\mathcal{O}(1)$ equipped with a metric. Before we look at that, here are a couple of points :

(1) Recall that if L is a holomorphic bundle, then there is a dual bundle L^*. It is defined set theoretically as $\cup_p L_p^*$ and the topology and complex structure are given by local trivialisations, i.e., if s_α are local holomorphic bases for L, then s_α^* defined as $s_\alpha^*(s_\alpha) = 1$ are local holomorphic bases of L^*. (So the transition functions are $\tilde{g}_{\alpha\beta} = 1/g_{\alpha\beta}$. (More generally, if E is a holomorphic (or even smooth for that matter) vector bundle, then $E^* = \cup_p E_p^*$ set theoretically. The topology and complex structure are given as : If $e_{\alpha,i}$ are local holomorphic bases for E then $e_\alpha^{*i}(e_{\alpha,j}) = \delta^i_j$ are holomorphic local bases of E^*. So the transition functions are $([g_{\alpha\beta}]^{-1})^T$.)

(2) If h is a Hermitian metric on L, then there is a natural Hermitian metric h^* on L^* defined as $h_\alpha^* = 1/h_\alpha$. Indeed, $h_\alpha^* = h_\beta^*|\tilde{g}_{\alpha\beta}|^2$ and hence h^* is a well-defined metric. Therefore, $c_1(h^*, L^*) = -c_1(h, L)$.

There is an obvious Hermitian metric on $\mathcal{O}(-1) \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ coming from the Euclidean metric on $\mathbb{C}^{n+1}$. Locally, in the chart U_0, if $s_0 \in \mathcal{O}(-1) = (1, z^1, z^2, \ldots)$ is a local basis, then $h_\alpha = 1 + |z|^2$. Hence, $c_1(\mathcal{O}(-1), h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\ln(1+|z|^2)$ and $c_1(\mathcal{O}(1), h^*) = \omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\ln(1+|z|^2)$. The associated Hermitian metric is $h_{i\bar{j}} = \frac{1}{\pi}\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\ln(1+|z|^2)$.

Given the Fubini-Study metric, we can calculate the induced metric on submanifolds. Indeed, let $F(X^i)$ be a degree d homogeneous polynomial such that $\nabla F \neq 0$ on $S = F^{-1}(0)$. As we saw earlier, S is a compact complex submanifold of $\mathbb{CP}^n$. Assume without loss of generality that $\frac{\partial F}{\partial X^1} \neq 0$

near a point on S where $X^0 \neq 0$. Therefore, locally, $z^1 = f(z^2,\dots,z^n)$ and $F(1, f, z^2, \dots) = 0$. Now $\omega_{FS}|_S = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\ln(1 + |f|^2 + \sum_{i=2}|z^i|^2)$, which equals

$$\frac{1}{1+|f|^2+\sum_{i=2}|z^i|^2}\frac{\sqrt{-1}}{2\pi}\left(\partial f \wedge \bar{\partial} f + \sum_{i=2} dz^i \wedge d\bar{z}^i - \frac{\sum_{i,j}(\bar{f}\frac{\partial f}{\partial z^i} + \bar{z}^i)(f\frac{\partial \bar{f}}{\partial \bar{z}^j} + z^j)dz^i \wedge d\bar{z}^j)}{1+|f|^2+\sum_{i=2}|z^i|^2}\right)$$

$$= \frac{1}{1+|f|^2+\sum_{i=2}|z^i|^2}\frac{\sqrt{-1}}{2\pi}\left(\delta_{i\bar{j}} + a_i\bar{a}_j - c_i\bar{c}_j\right)dz^i \wedge d\bar{z}^j \tag{1.1}$$

Noting that $\omega^n = \det(h_{i\bar{j}})n!\frac{\sqrt{-1}}{2}dz^1 \wedge d\bar{z}^1 \dots$, we can calculate the volume form by computing the determinant of the above matrix.

Exercise : Calculate the determinant of the matrix above.

Before we go on further, we need another notion from vector bundles. If E is a rank-r vector bundle, then $\Lambda^r E$ is also a vector bundle where $\Lambda^r_p E_p = E_p \wedge E_p \wedge \dots$. If $e_{\alpha,i}$ is a local basis of E, then $\eta_\alpha = e_{\alpha,1} \wedge e_{\alpha,2} \dots e_{\alpha,r}$ is a local basis for $\Lambda^r E$. If $e_{\beta,i} = [g_{\alpha\beta}]^j_{\ i} e_{\alpha,j}$, then

Exercise : Prove that $\eta_\beta = \det(g_{\alpha\beta})\eta_\alpha$

Therefore, if H is a Hermitian metric on E, then $\det(H)$ is a Hermitian metric on $\det(E)$. In particular, $\det(T^{*1,0}M) = K_M$ is called the canonical bundle of M and if ω is a Kähler form on M, then $\det(h)^{-1}$ (which is basically $(\frac{\omega^n}{n!})^{-1}$) is a Hermitian metric on K_M. Thus, $c_1(K_M, \det(h)) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\ln(\det(h))$.

Exercise : Compute $c_1(K_{\mathbb{CP}^n})$ with the metric induced from the Fubini-Study metric. A harder exercise is to compute $c_1(K_S)$ with the metric above. If you do the calculations correctly, you should get something like $c_1(K^n_{\mathbb{CP}}) = -(n+1)\omega_{FS}$, and $c_1(K_S) = (d-n-1)\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\psi$ for some smooth globally defined function ψ.

The above exercises show that $K_{\mathbb{CP}^n}$ is a negative line bundle, and if d is large, then K_S is a positive line bundle (and hence K^*_S is negative). When $d = n+1$, $[c_1(K_S)] = [0]$. Such an S (for example $(X^0)^5 + \dots (X^4)^5 = 0$ called the Fermat quintic) is called a Calabi-Yau manifold. These manifolds are important in String theory. They have nice Kähler metrics with good curvature properties.

2. Lecture 6 (Kähler connection and curvature)

Now we shall study the curvature of the Levi-Civita connection of Kähler manifolds. Recall that the Levi-Civita connection ∇ gives us a way to find the directional derivative of vector fields. $\nabla_X Y$ is the derivative of Y along X. It is uniquely determined by a few properties. More generally, given a smooth vector bundle E on a smooth manifold M, a connection is way to find the directional derivative of sections. It is defined as a map :

$$\nabla : \textit{Smooth sections of } E \times \textit{Smooth vector fields on } M \rightarrow \textit{Smooth sections of } E$$

satisfying

(1) $\nabla_{f_1X_1+f_2X_2}s = f_1\nabla_{X_1}s + f_2\nabla_{X_2}s$ where f_1, f_2 are smooth functions on M.
(2) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$.

(3) $\nabla_X(fs) = X(f)s + f\nabla_X s$.

If there is a Hermitian metric H on E, then ∇ is said to be metric compatible if $X(H(s_1, s_2)) = H(\nabla_X s_1, s_2) + H(s_1, \nabla_X s_2)$.

The Levi-Civita connection can be used to differentiate not just vector fields, but also induces a connection on T^*M by $X(\omega(Y)) = \nabla_X\omega(Y) + \omega(\nabla_X Y)$. The Christoffel symbols are defined as $\nabla_{\partial_k}\partial_j = \Gamma^i_{jk}\partial_i$. For the Levi-Civita connection there is a nasty formula for these beasts. But it is much simpler for calculations to note that the Christoffel symbols at p vanish in normal coordinates near p. For one-forms, $\nabla_{\partial_k}dx^j(\partial_i) = \partial_k(\delta^j_i) - \delta^j_l\Gamma^l_{ki}$ and hence $\nabla_{\partial_k}dx^j = -\Gamma^j_{ki}dx^i$. Using these two connections, we can talk about differentiating other tensors. For instance if $J = J^i_j dx^j \otimes \partial_i$, then $\nabla_X J$ is defined to be $\nabla_X J = X(J^i_j)dx^j \otimes \partial_i - J^i_j\Gamma^j_{kl}X^k dx^l \otimes \partial_i + J^i_j dx^j \otimes \Gamma^k_{il}X^l\partial_k$.

The Levi-Civita connection ∇ on a complex manifold can be extended complex linearly to a connection on $\mathbb{C}TM$. On a Kähler manifold, since there are holomorphic normal coordinates at every point p, $\nabla J = 0$ (because J has constant coefficients). We can define the Christoffel symbols in the $z, \bar{z}$ basis as follows.

$$\nabla_{\partial_{z^k}}\partial_{z^j} = \Gamma^i_{jk}\partial_{z^i} + \Gamma^{\bar{i}}_{jk}\partial_{\bar{z}^i} \tag{2.1}$$

and so on. Since $J\partial_{z^i} = \sqrt{-1}\partial_{z^i}$ and $\nabla J = 0$, we see that $J\nabla\partial_{z^i} = \sqrt{-1}\nabla\partial_{z^i}$. Therefore, $\nabla_{\partial_{z^k}}\partial_{z^i} \in T^{1,0}$ and hence $\Gamma^{\bar{i}}_{jk} = 0$. Now the torsion-free ness forces $\nabla_{\bar{i}}\partial_k = \nabla_k\partial_{\bar{i}}$ and hence both of them vanish. So the only surviving symbols are $\Gamma^i_{jk} = \Gamma^i_{kj}$ and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}$. One can now calculate the Christoffel symbols. The Levi-Civita connection satisfies $\nabla g = 0$. Since $\nabla J = 0$, $\nabla h = 0$.

$$\begin{aligned} h_{i\bar{j},k} &= \partial_k(h(\partial_i, \partial_{\bar{j}})) = h(\nabla_k\partial_i, \partial_{\bar{j}}) + h(\partial_i, \nabla_k\partial_{\bar{j}}) \\ &= h(\Gamma^l_{ki}\partial_l, \partial_{\bar{j}}) = \Gamma^l_{ki}h_{l\bar{j}} \\ &\Rightarrow \Gamma^i_{jk} = h^{i\bar{l}}\partial_j h_{k\bar{l}}. \end{aligned} \tag{2.2}$$

The upper indices indicate the inverse. So the matrix of 1-forms $\Gamma_k^{\,i} = [\partial h h^{-1}]_k^{\,i}$. The Riemann curvature tensor (extended $\mathbb{C}$-linearly) is

$$(\nabla_k\nabla_{\bar{l}} - \nabla_{\bar{l}}\nabla_k)\partial_i = R_{i\,k\bar{l}}^{\,j}\partial_j. \tag{2.3}$$

Exercise : Prove that the other covariant derivatives commute.

Upon computing

$$\begin{aligned} (\nabla_k\nabla_{\bar{l}} - \nabla_{\bar{l}}\nabla_k)\partial_i &= -\nabla_{\bar{l}}\Gamma^j_{ki}\partial_j \\ &= -\partial_{\bar{l}}(\Gamma^j_{ki})\partial_j = -\partial_{\bar{l}}(h^{j\bar{l}}\partial_k h_{i\bar{l}}). \end{aligned} \tag{2.4}$$

In holomorphic normal coordinates, $R_{i\,k\bar{l}}^{\,j}(p) = -\partial_{\bar{l}}\partial_k h_{i\bar{j}}(p)$. By the local $\partial\bar{\partial}$ lemma, since $h_{i\bar{j}} = \partial_{\bar{j}}\partial_i\phi$ for some smooth ϕ, we can interchange the derivatives to get many symmetries of the holomorphic Riemann curvature tensor. Recall that the Ricci curvature in usual Riemannian geometry is defined as $Ricc(Y, Z) = tr(X \to R(X, Y)Z)$. The Ricci tensor is symmetric.

In holomorphic normal coordinates, the Ricci tensor can be computed (extended $\mathbb{C}$-linearly) as

$$
\begin{aligned}
Ricc_{i\bar{l}} &= R^{\ \ j}_{i\ j\bar{l}}(p) = -\sum_j \partial_{\bar{l}}\partial_j h_{i\bar{j}}(p)\\
&= -\sum_j \partial_{\bar{l}}\partial_j\partial_i\partial_{\bar{j}}\phi(p) = -\sum_j \partial_{\bar{j}}\partial_j\partial_i\partial_{\bar{l}}\phi(p) = -\sum_j \partial_{\bar{l}}\partial_i h_{j\bar{j}}(p)\\
&= -\partial_{\bar{l}}\partial_i \ln(\det(h))(p). \qquad (2.5)
\end{aligned}
$$

The above calculation shows that $Ricc(JY, JZ) = Ricc(Y, Z)$. Akin to ω, we define a Ricci form on a Kähler manifold as $Ricc(JX, Y)$. It is clear that this form is a $(1, 1)$-form that is real (we will abuse notation and call this Ricci form also as the Ricci curvature sometimes). So the Ricci form is $\sqrt{-1}\bar{\partial}\partial\ln(\det(h))$. In other words it is simply $2\pi c_1(K_M^*)$.

Let us now connect the Ricci tensor in these complex coordinates (that is acting in a Hermitian manner on $T^{1,0}M$) to real coordinates. In almost the same way as h is related to g, the tensor $T(u, v) = Re(Ricc(Lu, \bar{L}v)) = Re(Ricc(\frac{u-\sqrt{-1}Ju}{2}, \frac{v+\sqrt{-1}Jv}{2}))$ is equal to $\frac{1}{2}Ricc(u, v)$. So the isomorphism gives a slightly different (by a factor of 2) Ricci tensor than the one we use in Riemannian geometry. The scalar curvature in Riemannian geometry is defined as the trace of the Ricci tensor.

Exercise : Prove that the scalar curvature in the complex setting above differs from the usual one by a factor of 4.

The Riemannian sectional curvatures are $g(R(u, v)v, u)$. In the Kähler case, calculations above show that $R(x, y, z, w) = R(x, y, Jz, Jw)$. For a unit vector x, we define the holomorphic sectional curvature as $H(x) = R(x, Jx, Jx, x)$ and for two orthonormal unit vectors, x, y, the bisectional curvature is $R(x, Jx, Jy, y)$.

Exercise : Show that the bisectional curvature is $R(x, Jx, Jy, y) = R(x, y, y, x) + R(x, Jy, Jy, x)$ *(thus justifying its name).*

Note that $R^{\ \ j}_{j\ i\bar{i}} = R(\partial_i, \partial_{\bar{i}}, \partial_j, \partial_{\bar{j}})$ is the following using the usual isomorphism between $T^{1,0}$ and TM (and symmetries of the Riemann tensor).

$$
R^{\ \ j}_{j\ i\bar{i}} = \frac{1}{4}R(\partial_{x^i}, J\partial_{x^i}, J\partial_{x^j}, \partial_{x^j}). \qquad (2.6)
$$

Exercise : Show that all holomorphic sectional curvatures of $\mathbb{C}^n$, $\mathbb{CP}^n$, *and* $\mathbb{D}$ *are 0, positive constants, and negative constants respectively.*

In the Riemannian case, when all the sectional curvatures are constant, the manifold is isometric to a quotient of Euclidean space, the Sphere, or Hyperbolic space. Akin to that, if all the holomorphic sectional curvatures are constant, the manifold is biholomorphically isometric to a quotient of $\mathbb{C}^n$, $\mathbb{CP}^n$, or $\mathbb{D}$.