ATM SCHOOL 2018 - SUBHARMONIC AND PLURISUBHARMONIC FUNCTIONS (LIBERALLY COPIED FROM KRANTZ'S BOOK, AND D. VAROLIN'S NOTES)

1. What is the end game ?

One of the aims of complex analysis is to classify domains upto biholomorphism. Another is to construct holomorphic functions with specified behaviour at given points. Both of these aims are more or less well studied in some avatars at least. (The Riemann mapping theorem and the Mittag-Leffler theorem.) In more than one complex variable, both problems are widely open and in fact, there is no analogue of the Riemann mapping theorem. (For instance, $B(0,1)$ is not biholomorphic to $D(0,1)^n$ when $n > 1$.)

It turns out that they key to understanding domains (whether in $\mathbb{C}$ or $\mathbb{C}^n$ for $n > 1$) and holomorphic functions on them is the study of certain PDE on them. For example, as we shall see later, one way to prove the Riemann mapping theorem in an important sub class of smoothly bounded (what does this mean?) domains is by solving the equation $\Delta u = 0$ on Ω with the Dirichlet boundary condition $u = f$ on the boundary (where f is continuous). In higher dimensions, a different, more complicated PDE (the complex Monge-Ampère equation) helps with the study of domains.

Prerequisites are multivariable calculus, linear algebra, and complex analysis.

2. What is a subharmonic function and how does one construct these beasts ?

Here are some observations :

(1) Harmonic functions in 1-D are linear $u(x) = ax + b$. Not too interesting. But, if we relax the requirement to $u'' \geq 0$, we get the rich class of convex functions. These obey the Jensen inequality : $tu(x_1) + (1-t)u(x_2) \geq u(tx_1 + (1-t)x_2)$. Moreover, their maximum is always attained on the endpoints.

(2) Harmonic functions in higher dimensions are also very rigid. (They obey the identity principle.)

Exercise : Prove that a C^2 solution $u : \Omega \subset \mathbb{C} \to \mathbb{R}$ of $\Delta u = 0$ is smooth.

Exercise : Prove the identity principle for harmonic functions $u : \Omega \subset \mathbb{C} \to \mathbb{R}$.

(Actually, the above exercises hold true in arbitrary dimensions.)

Harmonic functions (in any dimension) also obey the mean value property (MVP): Suppose $B(0,r) \subset \Omega$. Then

$$u(0) = \frac{\int_{\partial B(0,r)} u dA}{Vol(\partial B(0,r))} = \frac{\int_{B(0,r)} u dV}{Vol(B(0,r))} \tag{2.1}$$

Proof. Let $f(r) = \frac{\int_{\partial B(0,r)} udA}{Vol(\partial B(0,r))}$. Then f is differentiable (by dominated convergence). Now

$$f(r) = \frac{\int_{\partial B(0,1)} ur^{n-1} d\sigma}{r^{n-1} Vol(\partial B(0,1))}$$

$$\Rightarrow f'(r) = \frac{\int_{\partial B(0,1)} \frac{\partial u}{\partial r} d\sigma}{Vol(\partial B(0,1))} = \frac{\int_{B(0,1)} \Delta u dV}{Vol(\partial B(0,1))} = 0 \tag{2.2}$$

Thus $f(r) = \lim_{r\to 0} f(r) = u(0)$. Multiplying by dr and integrating, we see the volume version of the MVP. ◻

Exercise : Prove that if a C^2 function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies the MVP then it is harmonic.

Exercise : Prove that if a harmonic function attains its maximum or minimum in the interior of a domain, then it is a constant. (The strong maximum principle.)

From the above observations, it makes sense to relax our class of objects to "subharmonic functions", i.e., roughly functions of the type $\Delta u \geq 0$. However, we want to consider non-smooth functions too. The prototypical example to keep in mind is $u(z) = \ln |z|^2 = \ln(x^2 + y^2)$. This is of course harmonic away from $z = 0$ (because it is locally the real part of a branch of $ln(z)$). Note that $\ln |z|^2$ can be approximated using the decreasing sequence $\ln(|z|^2 + \frac{1}{n^2})$. So we instead of continuity, we want a property that is preserved under decreasing limits of functions. Moreover, we want the global maximum of the function on a compact set to make sense. The correct property we are looking for is upper semicontinuity, i.e., near x_0, the values of u should either be close to $u(x_0)$ or less than it. More precisely, $f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is called upper semicontinuous if $\limsup_{y\to x_0} f(y) \leq f(x_0)$ or equivalently, $u^{-1}[-\infty, y)$ should be open. Hence upper semicontinuous functions are Lebesgue measurable.

Exercise : Prove the equivalence of definitions of upper-semicontinuity. Also prove that upper semicontinuous functions achieve their maximum on a compact set , that if u_1, u_2 are upper semicontinuous then so is $u = \max(u_1, u_2)$, and that if $u_i, i \in I$ is a family of upper semicontinuous functions, then $\inf_{i\in I} u_i(x) = u(x)$ is upper semicontinuous.

Note that if u is upper semicontinuous (u.s.c), then u^+ is bounded and hence integrable. But u^- can have infinite integral. Thus $\int udV$ makes sense if we allow $-\infty$ as an answer. It is easy to see that so does the integral of u over the surface of a sphere make sense. We also have the following result.

Theorem 2.1. *If f is a u.s.c function on Ω that is bounded above, then there is a sequence of continuous functions $C \geq f_1 \geq f_2 \geq f_3 \ldots$ on Ω that converge to f.*

Proof. We give a sketch : The functions $f_j(x) = \sup_{y\in\Omega}\{f(y) - j|x - y|\}$ work. ◻

Exercise : Fill in the details above.

Definition 2.2. A subharmonic function $u : \Omega \subset \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is an upper-semicontinuous function that is not identically $-\infty$ and satisfies the following property : Suppose $K \subset \Omega$ is a compact set, and

h is a continuous function on K that is harmonic on its interior such that on ∂K, $u \leq h$, then $u \leq h$ on all of K.

Examples :

(1) A Harmonic function u is an example because for any compact set and a harmonic function h, $u - h$ is harmonic and we may use the maximum principle.
(2) A fundamental example is $\ln(|f|^2)$ where $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is a non-zero holomorphic function. Indeed, this is obviously upper-semicontinuous and is harmonic away from 0. If K contains 0 in its boundary, then $u - h$ is harmonic in the interior and hence cannot attain its maximum there. If K contain a zero of f in its interior, then we can reduce the problem to a disc centred at that zero of radius r by the previous argument. Taking $r \to 0$, we see that $u(z_0) = -\infty < h(z_0)$.
(3) If u is subharmonic and $c > 0$ is a constant, then cu is obviously subharmonic.
(4) If $u = \sup u_\alpha$ where u_α are subharmonic is finite and u.s.c, then u is subharmonic. So the maximum of two subharmonic functions is subharmonic.
(5) If $u_1 \geq u_2 \geq u_3 \ldots$ are subharmonic and their limit u is not identically $-\infty$ then u is subharmonic.

Here are some equivalent definitions.

Theorem 2.3. *Let $\Omega \subset \mathbb{C}^n$ be an open connected subset and let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ be u.s.c. TFAE*

(1) *u is subharmonic.*
(2) *If $\delta > 0$, $\bar{D}(z, \delta) \subset \Omega$, and μ is a non-negative Borel measure on $[0, \delta]$ with non-zero mass, then u satisfies the μ-SMVP :*

$$u(z) \leq \frac{\int_0^{2\pi} \int_0^\delta u(z + re^{i\theta}) d\mu(r) d\theta}{2\pi \int_0^\delta d\mu(r)} \tag{2.3}$$

(3) *For each $z_0 \in \Omega$, there exists a $\delta_{z_0} > 0$ such that $\bar{D}(z_0, \delta_{z_0}) \subset \Omega$ and for all $r \leq \delta_{z_0}$, and μ_{z_0} - a non-negative Borel measure on $[0, \delta_{z_0}]$ not supported on 0 with non-zero mass such that u satisfies the μ-SMVP on $D_r(z_0)$.*