ATM SCHOOL 2018 - SUBHARMONIC AND PLURISUBHARMONIC FUNCTIONS (LIBERALLY COPIED FROM KRANTZ'S BOOK, AND D. VAROLIN'S NOTES)

1. Recap

(1) Agreed that it is important to study the Dirichlet problem for $\Delta u = 0$ to prove the RMT.
(2) Proved the MVT for harmonic functions and discussed that they are too rigid to construct easily.
(3) Defined subharmonic functions and gave several examples.

2. What is a subharmonic function and how does one construct these beasts ?

Here are some equivalent definitions.

Theorem 2.1. *Let $\Omega \subset \mathbb{C}^n$ be an open connected subset and let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ be u.s.c. TFAE*

(1) *u is subharmonic.*
(2) *If $\delta > 0$, $\bar{D}(z, \delta) \subset \Omega$, and μ is a non-negative Borel measure on $[0, \delta]$ with non-zero mass, then u satisfies the μ-SMVP :*

$$u(z) \leq \frac{\int_0^{2\pi} \int_0^{\delta} u(z + re^{i\theta}) d\mu(r) d\theta}{2\pi \int_0^{\delta} d\mu(r)} \tag{2.1}$$

(3) *For each $z_0 \in \Omega$, there exists a $\delta_{z_0} > 0$ such that $\bar{D}(z_0, \delta_{z_0}) \subset \Omega$ and for all $r \leq \delta_{z_0}$, and μ_{z_0} - a non-negative Borel measure on $[0, \delta_{z_0}]$ not supported on 0 with non-zero mass such that u satisfies the μ-SMVP on $D_r(z_0)$.*

Proof. (1) 1 implies 3 : Choose a decreasing sequence f_j of continuous functions converging to u on $\bar{D}$. Solve $\Delta h_j = 0$ on D with $h_j = f_j$ on ∂D. This can be done by an explicit formula. (See Poisson kernel on wikipedia or in Krantz's book.) Since $u \leq h_j = f_j$ on ∂D, by definition $u(z) \leq h_j(z) = \frac{\int_0^{2pi} f_j(z+re^{i\theta}) d\theta}{2\pi}$. By the monotone convergence theorem we are done.
(2) 2 implies 3 : Simply integrate on both sides w.r.t measure.
(3) 3 obviously implies 4.
(4) 4 implies 1 : Suppose $u \leq h$ on ∂K. Let M be the maximum of $v = u - h$ on K. If 1 is not true, then for some K and some h, $M > 0$ at a point p in the interior of K. The set $F \subset K$ where $v = M$ does not meet ∂K. Let $z_0 \in F$ have minimal positive distance from K and let $\delta_{z_0} > 0$ be less than this distance. Then the μ-SMV property provides a contradiction.

□

Exercise : Let u be subharmonic and $p \in \Omega$. Prove that the averages of u over circles of centre z_0 and radii r converge to $u(z_0)$ as $r \to 0$.

This has the following consequences

Corollary 2.2. (1) *$u_1 + u_2$ is subharmonic if u_1, u_2 are so.*
(2) *Subharmonicity is a local property, i.e., u is subharmonic in Ω iff it is locally so.*
(3) *If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and increasing, then $\phi \circ u$ is subharmonic whenever u is so.*

(4) *If the maximum of a subharmonic function u over a bounded connected open set Ω is attained in the interior, then u is constant on Ω.*
(5) *Subharmonic functions are locally integrable.*

Proof. (1) The μ-SMV property obviously holds for the sum if it holds individually.
(2) Follows from property 4 above.
(3) Follows from the Jensen inequality $\phi(< u >) \leq < \phi(u) >$ and property 4 above.
(4) Let M be the maximum. $u^{-1}(M)$ is closed by upper semicontinuity. It is also open by the μ-SMVP.
(5) Suppose X is the set of $z \in \Omega$ such that u is locally integrable over a small disc. Obviously X is open. It is non-empty because if $u(a) > -\infty$ (which happens at least for one a by assumption), by the SMVP the average over a small disc centred at a is $> -\infty$. X is also closed (and hence all of Ω) because if $p_n \in X \to p$ (and $p \in \Omega$), then choosing a small disc around p which lies wholly in Ω, clearly it contains a close enough p_n and a smaller disc centred at p_n (containing p). By the SMVP u is locally integrable on this disc.

□

Finally, we have a very useful characterisation of subharmonic functions in terms of distributions. We say that for a locally integrable u, $\Delta u \geq 0$ in the sense of distributions iff $\int_{\Omega} u\Delta\phi \geq 0$ for any smooth function $\phi \geq 0$ with compact support in Ω. We prove the following alternate characterisation of subharmonic functions.

Theorem 2.3. *If u is subharmonic, then $\Delta u \geq 0$ in the sense of distributions. Conversely, if f is locally integrable and $\Delta f \geq 0$ in the sense of distributions then it can be modified on a set of measure 0 to become subharmonic.*

Exercise : Prove the above for smooth functions (Hint : Prove/Use the SMVP).

Now we recall an important technical device of smoothing out locally integrable functions. Suppose $\psi(x) = \psi(|x|) \geq 0$ is a smooth function that is compactly supported in the unit ball and has integral 1. For $\epsilon > 0$, define $\psi_\epsilon = \frac{1}{\epsilon^n}\psi(x/\epsilon)$. Now for x in the interior of Ω, for sufficiently small ϵ (such that the following integral makes sense) $u_\epsilon(x) = u * \psi_\epsilon(x) = \int_{B(0,\epsilon)} u(x-y)\psi_\epsilon(y)dy = \int_{B(x,\epsilon)} u(y)\psi_\epsilon(x-y)dy$ is smooth in x when u is locally integrable. In fact,

Lemma 2.4. (1) Exercise : If u is continuous, then $u_\epsilon \to u$ on compact subsets of Ω.
(2) *If $u \in L^p$ where $1 \leq p < \infty$ then $u_\epsilon \to u$ in L^p.*

The proof is in the appendix of Evans' book.
Now we have

Lemma 2.5. *If u is subharmonic and smooth, then u_ϵ are subharmonic and decrease to u pointwise as $\epsilon \to 0$.*

Proof. Indeed, $\Delta u_\epsilon = (\Delta u) * \psi_\epsilon \geq 0$ and hence they are subharmonic. $u_\epsilon(x) = \int_{B(0,\epsilon)} u(x - re^{i\theta})d\theta\psi_\epsilon(r)rdr$ which we know is a decreasing function of ϵ. □

Finally we prove the theorem above :

If u is subharmonic,

$$\int_{B(z_0,r)} u_\epsilon dV = \int_{B(z_0,r)} \int_{\mathbb{R}^N} u(x-\epsilon t)\psi(t)dtdx$$

$$= \int_{\mathbb{R}^N} \int_{B(z_0,r)} u(x-\epsilon t)\psi(t)dxdt \geq Vol(B(z_0,r)) \int_{\mathbb{R}^N} u(z_0-\epsilon t)\psi(t)dt \tag{2.2}$$

and hence the SMVP holds and by an above result the smooth function u_ϵ satisfies $\Delta u_\epsilon \geq 0$. Also, $u_\epsilon \to u$ pointwise a.e. Indeed, if $u_\epsilon \to v$ pointwise, and $v > u$ on a set of non-zero measure, then $\lim \int u_\epsilon = \int v > \int u$ which is a contradiction because $u \in L^1_{loc}$ and hence the above results say that $u_\epsilon \to u$ in L^1_{loc}.
Hence if ϕ has compact support, $\int u\Delta\phi = \lim_{\epsilon\to 0} u_\epsilon \Delta\phi \geq 0$.

If $\Delta f \geq 0$, then f_ϵ is smooth and satisfies $\Delta f_\epsilon \geq 0$ (because $\Delta f \geq 0$ and $\psi \geq 0$). It is decreasing in ϵ. Indeed, double smooth f by taking $(f_\epsilon)_\delta) = (f_\delta)_\epsilon$ which is of course decreasing in ϵ for every fixed δ. Letting $\delta > 0$, we see that f_ϵ is subharmonic and decreasing in ϵ. Therefore the limit g is subharmonic and is easily seen to satisfy (by the μ SMVP) that $\int (f-g)\phi = 0$ for all smooth compactly supported ϕ. Hence $f = g$ almost everywhere. □

Actually, the above technique allows us to prove Weyl's lemma

Lemma 2.6. *If a locally integrable function u satisfies $\Delta u = 0$ in the sense of distributions, then it is smooth.*

Proof. Indeed, $\int u(y)\Delta_y \psi_\epsilon(x-y)dy = 0$ which means that $\Delta_x u_\epsilon(x) = 0$. By the μ-SMVP, $u_\epsilon = u$ which is smooth. □