MA 200 - Lecture 19

1 Recap

1. Completed proof of Lebesgue's theorem.
2. Proved Fubini.
3. Defined integrals over bounded sets (and proved well-definedness).

2 Integral over a bounded set

Let $f, g : S \subset \mathbb{R}^n \to \mathbb{R}$ be two bounded functions. If f, g are continuous at $a \in S$, then it is easy to see that $\max(f, g)$ and $\min(f, g)$ are so too. Likewise, if f, g are Riemann-integrable over S, using Lebesgue's theorem it is easy to see that $\max(f, g)$ and $\min(f, g)$ are Riemann integrable over S. The Riemann integral satisfies a number of familiar properties:

1. Linearity ($\int_S (af + bg)$ exists if $\int_S f$ and $\int_S g$ exist and equals their linear combination): We can assume WLOG that S is a rectangle. On it, by Lebesgue, we see that $af + bg$ is R.I. Assume first that $a, b \geq 0$. For any partition, by finite-linearity, $\int_S (af + bg)$ lies between $L(P, f)$ and $U(P, f)$. So does $a \int f + b \int g$. By refining partitions, we are done in this case. If we prove that $-\int f = \int(-f)$, we are done in general. Indeed, a simple partition argument does the job.

2. Comparison (If $f \leq g$, then $\int f \leq \int g$) and estimation $|\int f| \leq \int |f|$): For rectangles it is easy. For $|f|$, note that $|f| = \max(f, -f)$.

3. Monotonicity (If $T \subset S$ and $f \geq 0$, then $\int_T f \leq \int_S f$): Note that $f_T \leq f_S$ and hence by comparison we are done.

4. Additivity (If $S = S_1 \cup S_2$, and f is R.I over S_1, S_2, then it is so over $S = S_1 \cup S_2, T = S_1 \cap S_2$ and $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$: Note that if $f \geq 0$, then $f_S = \max(f_{S_1}, f_{S_2})$ and $f_T = \min(f_{S_1}, f_{S_2})$. So these are R.I. If f is not non-negative, then $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ are R.I and moreover $f = f_+ - f_-$. So we are done. The additivity formula follows from $f_S = f_{S_1} + f_{S_2} - f_T$.

As a corollary, if $S_i \cap S_i$ has measure zero for all $i \neq j$, then $\int_S f = \int_{S_1} f + \int_{S_2} f + \ldots$. When is a continuous function f R.I over a bounded set S? The answer is provided by the following theorem.

Theorem 1. *Let $S \subset \mathbb{R}^n$ be a bounded set and $f : S \to \mathbb{R}$ be a bounded continuous function. Let $E \subset Bd(S)$ be the set of points x_0 such that $\lim_{x\to x_0} f(x) = 0$ fails to hold. Then if E has measure 0, f is R.I over S. (In particular, if $Bd(S)$ has measure zero - such domains are called rectifiable, a continuous bounded function is R.I over S.)*

Proof. Note that if $x \in E^c$, then either $x \in Int(S)$ (in which case x is a point of continuity of f) or $x \in Bd(S)$ but $\lim_{x\to x_0} f(x) = 0$ where x approaches from points in S. Now $f_S(x_0) = 0$ if $x_0 \in S$ (by continuity) or if x_0 is outside S by definition. Hence, for such points, $|f(x)| < \epsilon$ whenever $|x - x_0| < \delta$ and $x \in S$ (by assumption) or $x \in S^c$ (by definition). □

Of course, coming up with rectifiable domains does not appear to be trivial. Fortunately, your HW gives you a way to do so (the unit disc for instance). Lastly, Lebesgue's theorem and similar reasoning as above shows that

Theorem 2. *If $f : S \to \mathbb{R}$ is bounded continuous (and S bounded), then if f is R.I over S, it is R.I over $Int(S)$ and the integrals are equal.*

3 Improper integrals

How can we define say $\int_0^1 \frac{1}{\sqrt{x}}dx$ or $\int_1^\infty \frac{1}{x^2}dx$? The easiest way is through limits. More generally,
Def: Let $A \subset \mathbb{R}^n$ be an open set (possibly unbounded) and $f : A \to \mathbb{R}$ be a continuous (not necessarily bounded) function. If $f \geq 0$ on A, we define the improper integral of f over A to be $\sup \int_D f$ where D ranges over compact rectifiable subsets of A provided this supremum is finite. In this case, we say that f is R.I over A in the improper sense. More generally, we say that f is R.I over A in the improper sense if f_+ and f_- are, and the improper integral is improper integral of f_+ minus that of f_-.
Are there enough compact rectifiable subsets at all in the first place? Thankfully yes:

Lemma 3.1. *Let $A \subset \mathbb{R}^n$ be open. Then there is a sequence C_N of compact rectifiable subsets such that $C_N \subset Int(C_{N+1})$ and $A = \cup_N C_N$. (Such a sequence is called a compact rectifiable exhaustion of A.)*

Proof. Let S_N be the set of all points in A that are at a distance of at least $\frac{1}{N}$ from the boundary of A (which is a closed set) and of distance at most N from the origin. Note that $A = \cup_N S_N$ (why?) and $S_N \subset Int(S_{N+1})$ (why?) Unfortunately, S_N need not be rectifiable. So we construct C_N by simply covering S_N by closed cubes that lie in the interior of C_{N+1}. We need only finitely many such cubes and their union is C_N. The boundary of C_N contains finitely many sets of measure zero and hence is of measure zero. Since $S_N \subset C_N$, the other properties are met. □

The following theorem can be proven by dividing into two cases ($f \geq 0$ and $f = f_+ - f_-$) and following one's nose.

Theorem 3. *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be continuous. Choose a compact rectifiable exhaustion C_N of A. Then f is improper R.I over A iff $\int_{C_N} |f|$ is bounded independent of N. In this case, $(Improper)\ \int_A f = \lim_{N\to\infty} \int_{C_N} f$.*

By passing to compact rectifiable exhaustions, one can prove linearity, comparison and estimation, monotonicity, and additivity.
If A is bounded, and f is bounded and continuous, does the improper integral coincide with the usual one? Thankfully, yes (if both integrals exist):

Theorem 4. *Let A be bounded and open, and $f : A \to \mathbb{R}$ be bounded and continuous. Then the improper integral exists. Suppose f is* also *R.I (this need not always happen). Then these integrals are equal.*

Proof. Since f is bounded and so is A, of course $\int_{C_N} |f|$ is bounded independent of N. Thus the improper integral exists. Now assume f is R.I. WLog $f \geq 0$. Indeed, if not, $f = f_+ - f_-$ where f_+, f_- are improper integrable and R.I (the former by definition and the latter by a theorem). Now by additivity of the usual and the improper integrals we reduce to the case where $f \geq 0$.
By monotonicity, if $D \subset A$ is compact and rectifiable, then $\int_D f \leq \int_A f$ in the usual sense. Since D is arbitrary, the improper integral is bounded above by the usual one.
Now let P be a partition of Q (containing A) and R_i be the subrectangles lying in A. Then if $D = \cup_i R_i$, we see that by properties (and the fact that $m_R(f_A) = m_R(f)$ if R is contained in A and $m_R(f_A) = 0 \leq m_R(f)$ otherwise), $L(f_A, P) \leq \int_D f \leq (Improper) \int_A f$. Since P is arbitrary, we are done. □