MA 200 - Lecture 27

1 Recap

1. Defined the wedge product using Alt.
2. Exterior derivative of form-fields.
3. Started pullback.

2 Exterior derivative and pullback

Def: Suppose $F : U \subset \mathbb{R}^k$ or $\mathbb{H}^k$ to $\mathbb{R}^n$ is a smooth map, and ω is a smooth $k \geq 1$-form on $V \subset \mathbb{R}^k$ or $\mathbb{H}^k$ such that $F(U) \subset V$. Then the *pullback* $F^*\omega$ is a smooth k-form on U defined as $F^*\omega$ at the point x is simply $DF^*\omega$ at the point $F(x)$, i.e., $(F^*\omega)(x)(v_1, \ldots, v_k) = \omega(f(x))(Dfv_1, Dfv_2, \ldots)$. For 0-forms, $F^*f(x) = f \circ F(x)$.
Since we have already proven properties of T^*, we see that $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ (this is true even if 0-form (why?)), F^* is linear, and $(G \circ F)^*\omega = F^* \circ G^*\omega$. Here are examples of pullbacks:

1. $\omega = dx \wedge dy$. Let $F(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Then $F^*\omega = F^*dx \wedge F^*dy$. Now $F^*dx(v, w) = dx(DF(v, w)) = dx(\frac{\partial r\cos(\theta)}{\partial r}v + \frac{\partial r\cos(\theta)}{\partial \theta}w, \ldots) = \cos(\theta)v - r\sin(\theta)w = d(r\cos(\theta))(v, w)$. Indeed, more generally, $F^*df = d(f \circ F)$ because $F^*df(v) = df(DFv) = \sum \frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial x_j}v_j = d(f \circ F)(v)$.
2. If $\gamma(t) = (\cos(t), \sin(t))$, then $\gamma^*dx = d(x \circ \gamma) = d(\cos(t)) = -\sin(t)dt$ as we expect.
3. More generally, $F^*(d\omega) = d(F^*\omega)$: Indeed, $F^*(\sum_i d\omega_I \wedge \epsilon_I) = \sum_i F^*(d\omega_I) \wedge F^*dx_{i_1} \wedge \ldots = \sum_i d(\omega_I \circ F) \wedge dF_{i_1} \ldots$. Now $d(F^*\omega) = d(\sum_I \omega_I \circ F dF_{i_1} \wedge \ldots) = \sum_I dF^*\omega_I \wedge dF_{i_1} \ldots + F^*\omega_I d(dF_{i_1}) \wedge \ldots + \ldots$ but $ddf = 0$ and hence we are done. $\square$
4. Suppose $y_1, \ldots, y_n$ are coordinates in $\mathbb{R}^n$ and I is an increasing multi-index of size k. Then $F^*(\epsilon_I)(e_1, \ldots, e_k) = \epsilon_I(DFe_1, \ldots, DFe_k) = \det(\frac{\partial(F_{i_1}, \ldots)}{\partial(x_1, \ldots, x_k)})$. In other words, $F^*\epsilon_I = \det(\frac{\partial(F_{i_1}, \ldots)}{\partial(x_1, \ldots, x_k)})dx_1 \wedge dx_2 \ldots dx_k$.

3 Integrating top forms in $\mathbb{R}^n$

Let ω be a smooth n-form field in an open subset $U \subset \mathbb{H}^n$ or $\mathbb{R}^n$. Then $\omega = f dx_1 \wedge dx_2 \ldots dx_n$. Define $\int_U \omega = \int_U f$ as an improper integral if it exists. Note that if it does exist, then this integral is indeed $\int_{Int(U)} f$. The point of the definition is the following: Suppose $\phi : V \to Int(U)$ is a smooth diffeomorphism (that is, a homeomorphism that is smooth and whose derivative is injective throughout), then by the change of variables formula, $\int_{Int(U)} f = \int_{Int(V)} f \circ \phi |\det(D\phi)|$. If ϕ is orientation-preserving, then $\int_U f = \int_V f \circ \phi \det(D\phi)$. However, recall that $\phi^*(dx_1 \wedge \ldots) = \det(D\phi) dy_1 \wedge \ldots$. Thus, $\int_U \omega = \int_V \phi^* \omega$ provided ϕ is orientation-preserving. Otherwise, $\int_U \omega = -\int_V \phi^* \omega$.

Stokes' theorem in $\mathbb{H}^n$: Let ω be a smooth $n-1$-form field with compact support in $\mathbb{H}^n$. Then $\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{x_n=0} i * (\omega)$, where $i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, 0)$.

Proof. Suppose $\omega = \omega_1 dx_2 \wedge dx_3 \ldots + \omega_2 dx_1 \wedge dx_3 + \ldots$. The integral is (using Fubini's theorem) $\int_0^a \int_{-a}^a \ldots \sum_i (-1)^{i-1} \partial_i \omega_i dx_1 dx_2 \ldots dx_n$. Using the fundamental theorem of calculus, we see that $\int_{-a}^a \partial_i \omega_i dx_i = 0$ whenever $1 \leq i \leq n-1$ because ω has compact support. For $i = n$, we get $\int_{-a}^a \ldots (-1)^n \omega_n(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots$. $\square$

The same proof shows that if ω has compact support in $\mathbb{R}^n$, then $\int_{\mathbb{R}^n} d\omega = 0$.

4 Differential forms on manifolds (with or without boundary)

Let $M \subset \mathbb{R}^n$ be a smooth k-manifold-with-boundary. A smooth k-form field on M is a map $\omega : M \to \cup_p \Lambda^k(T_p M)$ such that $\omega(p) \in \Lambda^k(T_p M)$ and for any smooth parametrisation α, the form field $\tilde{\omega}(x)(v_1, \ldots, v_k) = \omega_{\alpha(x)}(D\alpha v_1, \ldots)$, i.e., $\alpha^* \omega$ is smooth. Clearly, if ω is a smooth k-form field in an open neighbourhood U of M in $\mathbb{R}^n$, then when restricted to M, it is a smooth form field (by the chain rule). It can be proven (but the proof is not trivial) that every smooth k-form field can be extended to a smooth k-form field in a neighbourhood of M. We shall assume this statement from now onwards (for the sake of simplicity). Here are examples of form-fields:

1. The 1-form field $\omega = \frac{xdy - ydx}{x^2+y^2}$ on the unit circle. If we choose the parametrisation of a part of the unit circle given by $\alpha(t) = (\cos(t), \sin(t))$, then $\alpha^*(\omega) = dt$.
2. The 2-form field $\omega = z^2 dx \wedge dy + x^2 dy \wedge dz$ on the unit upper hemisphere. If we parametrise a part of it using $(x, y, \sqrt{1-x^2-y^2})$, then $\alpha^* \omega = (1 - x^2 - y^2 + \frac{x^3}{\sqrt{1-x^2-y^2}}) dx \wedge dy$.

The next step is to define the exterior derivative. Since we are assuming that smooth form fields are actually restrictions of those appearing from an open neighbourhood, we simply take the restriction of $d\omega$. We have the following result that directly follows from the properties of the usual exterior derivative.

Lemma 4.1. *Let M be a smooth k-manifold-with-boundary. For every coordinate parametrisation α, $d(\alpha^* \omega) = \alpha^*(d\omega)$ where $d(\alpha^* \omega)$ is the exterior derivative in $\mathbb{R}^k$.*