# MA 200 - Lecture 9

## 1 Recap

1. Proved IFT for one-variable functions and recognised the key step.
2. Started it for several variables.

*Proof.* We first prove for $r = 1$:

1. $f$ is locally $1 - 1$.
2. The image of a neighbourhood of $f$ is open: We wish to prove that every point near $f(a)$ lies in the image of $f$, i.e., the image of $f$ contains an open ball $B_{f(a)}(r')$. Then $f^{-1}(B_{f(a)}(r')) \cap B_a(r) = U$ will be the desired neighbourhood whose image is open (and one where $f$ is $1 - 1$). That is, we want to produce an $r' > 0$ such that for every $b \in B_{f(a)}(r')$, we can solve $f(x) = b$ to get an $x$. There are two ways of doing this (one of them is the usual way and the other is in the textbook):

   (a) Iteration/contraction mapping principle: $x_{n+1} = x_n - Df(x_n)^{-1}(f(x_n) - b)$ if the later makes sense. Firstly, recall that on $B_a(r)$, $\|(Df)^{-1}\| \leq C$. If we choose $r'$ (which is $> \|f(a) - b\|$) to be small enough, we claim that all the $x_n$ belong to $B_a(r'')$ where $r'' < r$ and that $f(x_n) \in B_b(r')$. The rough idea is that $f(x_{n+1}) - b \approx (f(x_n) - b) - (f(x_n) - b) = 0$ and hence less than $r'$. (Moreover, $x_{n+1} - x_n \approx 0$ and hence the geometric series sum will show that $x_n$ is close to $a$ for all $n$.)
   Firstly, there exists $r'' < r$ and (how? - HW exercise) $y, z \in B_a(r'')$, we have $\|Df(y)(Df)^{-1}(z) - I\|_{Frobenius} < \epsilon = \frac{1}{2}$. (The choice of this $\epsilon$ comes from hindsight rather than foresight.) In fact, something stronger is true: One can choose $r'$ so that if $y_1, y_2, \ldots, y_n, z_1, \ldots, z_n \in B_a(r')$ then $\|AB^{-1} - I\|_{frob} < \frac{1}{2}$, where the $i^{th}$ row of $A$ is $\nabla f_i(y_i)$ and that of $B$ is $\nabla f_i(z_i)$.
   Secondly, $x_2 = a - (Df(a))^{-1}(f(a) - b)$ and hence $\|x_2 - a\| \leq C\|f(a) - b\| < Cr' < r''$ if $r' = \frac{r''}{4C}$ (again from hindsight). (Stopped here the last time.)

   We shall inductively prove that $\|x_{n+1} - x_n\| \leq \frac{r''}{2^{n+1}}$ and $\|f(x_n) - b\| < \frac{1}{2}\|f(x_{n-1} - b)$: Moreover, by the mean-value-theorem applied to each component of $f(x)$, $f_i(x_2) = f_i(a) - \langle (\nabla f_i)_{\theta_{i,1}a + (1-\theta_{i,1})x_2}, (Df(a))^{-1}(f(a) - b)\rangle$. Thus (how again?) $\|f(x_2) - b\| < \epsilon\|f(a) - b\|$. Assume inductively for $i = 1, 2, \ldots, n$ that $\|f(x_i) - b\| < \epsilon\|f(x_{i-1}) - b\|$, and that $x_i \in B_a(r'')$.
   Now for $i = n + 1$, $\|x_{n+1} - x_n\| \leq C\|f(x_n) - b\| \leq C\epsilon^{n-1}\|f(a) - b\|$. Thus,

$\|x_{n+1} - a\| \leq 2C\|f(a) - b\| < 2Cr' < r''$. Thus $x_{n+1} \in B_a(r'')$. Now the same mean-value-trick as in the base case shows that $\|f(x_{n+1}) - b\| < \epsilon\|f(x_n) - b\|$. To summarise, we have shown that if $\|f(a) - b\| < r' = \frac{r''}{4C}$, then $\|x_{n+1} - x_n\| \leq \frac{r''}{2^{n+1}}$ and that $\|f(x_n) - b\| < \frac{r'}{2^{n-1}}$. Thus this sequence is Cauchy (why?) and hence converges to some $x_*$. Since $f$ is continuous, $f(x_n) \to f(x_*)$. Now $\|f(x_n) - b\|$ converges to $0$ and hence $f(x_*) = b$.
We can also phrase this entire business in terms of the contraction mapping principle applied to the function $g(x) = x - (Df_a)^{-1}(f(x) - b)$ (from a closed ball around $a$ to itself. This is in Rudin's book).

(b) Maxima/Minima (in the text): First we need an elementary result: Let $\phi : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable on the open set $U$. If has a local with $-\phi$, this works for local maxima too) at $x_0$ (by the way, $U$ does not have to be open. All we need is for $x_0$ to be an interior point), then $D\phi_{x_0} = 0$. Indeed, consider the one-variable function $g(t) = \phi(x_0 + tv)$ on $t \in (-\epsilon, \epsilon)$. This function has a local min at $t = 0$ (why?) and hence by one-variable calculus, $g'(0) = 0 = \nabla_v \phi(x_0)$. Since this is true for all $v$, we are done.
Given $f(a)$, we want to show that $B_{f(a)}(r')$ is in the image for some $r' > 0$, i.e., given $c \in B_{f(a)}$, we want to find $x' \in B_a(r)$ such that $f(x') = c$. The idea is to minimise the function $g(x) = \|f(x) - c\|^2$ over an appropriate domain $Q$ and show that the minimum is $0$. Indeed, choose a closed rectangle (or if you like a closed ball) that contains $a$ and lies entirely within $B_a(r)$ and such that $Df_x$ is invertible when $x \in Q$ (how is this possible?). Then $Q$ is compact and hence $g(x)$ does attain a minimum. Choose and $r'$ so that the ball $B_{f(a)}(2r')$ does not intersect $f(Bd(Q))$ (why is this possible? because $f$ is $1-1$). Now the minimum of $g$ cannot be attained on $Bd(Q)$ (why?) Hence it is attained at an interior point $x'$. By the above, $\nabla g(x') = 0$ which means that $Df_{x'}(f(x) - c) = 0$. By invertibility, $f(x) = c$.

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