MA 200 - Lecture 21

1 Recap

1. Examples of improper integrals.
2. Existence of partitions-of-unity subordinated to a given open cover.
3. Improper integrals as sums of series of proper ones.

2 Change of variables

We need an analogue of the technique of substitution in multivariable calculus. A change of variables is simply a C^r ($r \geq 1$) diffeomorphism $g : A \to B$ where A, B are open sets. As a consequence, $\det(Dg) \neq 0$ everywhere.

Theorem 1. *Let $g : A \to B$ be a change of variables. Let $f : B \to \mathbb{R}$ be a continuous function. Then f is improper integrable over B iff $f \circ g|\det(Dg)|$ is improper integrable over A. In this case, $\int_B f = \int_A (f \circ g)|\det(Dg)|$.*

Before we prove the theorem, here are some examples:

1. For $n = 1$, Let $A = [a, b]$ (where $a < b$) and $B = [c, d]$ (where $c < d$). Then $\int_B f(y)dy = \int_A f(y(x))|y'|dx$. But in usual substitution, there is no $|.|$. How to resolve this discrepancy? The point is that $\int_B f(y)dy = \int_c^d f(y)dy$ and $\int_A h(x)dx = \int_a^b h(x)dx$. If $y' < 0$ somewhere, it is so *everywhere* and hence $y(a) = d$ and $y(b) = c$. Thus, $\int_a^b f(y(x))|y'|dx = -\int_a^b f(y(x))y'(x)dx = \int_b^a f(y(x))y'(x)dx = \int_c^d f(y)dy$.
2. Find $\int_{B_0(1)} x^2y^2dA$: Of course this function is R.I. We would like to use polar coordinates but $g(r, \theta) : A = (0, 1) \times (0, 2\pi) \to \mathbb{R}^2$ is a diffeomorphism, NOT to the entire unit disc but to the unit disc minus the non-negative x-axis (it is of course $1 - 1$, onto, and smooth. The point is that by IFT, since $\det(Dg) = r \neq 0$, the local inverse (which coincides with g^{-1}) is smooth locally. However, happily enough, this set has measure zero and hence $\int_{B_0(1)} x^2y^2dA = \int_{g(A)} x^2y^2$ (indeed, we proved that if f vanishes on a set of measure zero, the R.I is 0. Hence if a R.I function coincides with another function upto measure zero, the other function is also R.I and has the same integral!) Hence the integral is (after change of variables and Fubini) $\int_0^{2\pi}\int_0^1 r^5 \cos^2(\theta)\sin^2(\theta)drd\theta$.

3. Consider the improper integral $\int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} dy$. It is *not* an ordinary R.I. However, $y = \sin(x)$ converts it into an ordinary one. So in the statement of change of variables, we really need the improper part. If both integrals exist in the ordinary sense, then they are equal.

One important observation: A change of variables takes compact rectifiable subsets of A to compact rectifiable subsets of B (and takes their interiors to interiors and boundaries to boundaries). This follows from general topology (a diffeo is a homeo in particular) and the fact that measure zero sets are taken to measure zero ones by C^1 maps (HW).
Now we prove the theorem.

Proof. Note that the "only if" part is enough to imply the "if" part: Indeed, simply consider $F = f \circ g|Dg|$, replace g by g^{-1}, and use the chain rule. So we shall only consider the "only if" part of the theorem from now onwards.
The idea of the proof is to write the diffeo g *locally* as a composition of simpler "primitive pieces" (Def: A diffeomorphism (where $n \geq 2$) $h = (h_1, \ldots, h_n)$ is called primitive if $h_i(x) = x_i$ for some i.) which are amenable to induction. This step involves IFT. We also need to prove the theorem for primitive diffeos, that the theorem behaves well under compositions, and that it is enough for this local decomposition to hold.

1. If the theorem holds for g_1, g_2, it does for $g_1 \circ g_2$: Indeed, $g_1 \circ g_2$ is a diffeo and $\int_W f = \int_V f \circ g_1 |(Dg_1)| = \int_U f \circ g_1 \circ g_2 |Dg_1 \circ g_2||Dg_2|$. Now note that if $h = g_1 \circ g_2$, $Dh = Dg_1 \circ g_2 Dg_2$ and the determinant is multiplicative.

2. Suppose for each $x \in A$, there is a neighbourhood $U_x \subset A$ such that the theorem holds for all continuous f whose supports are compact and lie in $V_x = g(U_x)$, then it holds for all g, f: Wlog $f \geq 0$. Now take a partition-of-unity ϕ_i of B subordinate to V_x (where x ranges over all of A). Then $\int_B f = \sum_i \int_B \rho_i f = \sum_i \int_{V_{x_i}} \rho_i f = \sum_i \int_{U_{x_i}} \rho_i \circ g f \circ g |\det(Dg)| = \sum_i \int_A \rho_i \circ g f \circ g |\det(Dg)|$. Now $\rho_i \circ g$ is a partition-of-unity of A (why?) Hence $\sum_i \int_A \rho_i \circ g f \circ g |\det(Dg)| = \textit{Improper} \int_A f \circ g |\det(Dg)|$.

3. The theorem holds for primitive diffeos g: We shall prove this by induction on n. For $n = 1$, we are done by substitution. Assume this statement for $1, 2 \ldots, n-1$. Assume that $h_n(t) = t$ wlog and that $h(t) = (k(x, y), y)$ where $x \in \mathbb{R}^{n-1}$. Let $Q \subset V$ be a rectangle containing $q = f(p), p \in U$. Let $S = h^{-1}(Q)$. Note that $h : Int(S) \to Int(Q)$ is a change of variables. By the previous steps, it is enough to assume that f is compactly supported in Q. Hence, $\int_B f = \int_Q f = \int_I \int_D f(x,t) dx dt$ by Fubini. Let $F = f \circ h |\det(Dh)|$ on S and 0 outside. (F is continuous.) Note that since $h_n(t) = t$, S is contained in $E \times I$ for some rectangle E. Therefore, we need to prove that $\int_I \int_D f(x,t) dx dt = \int_I \int_E F(y,t) dy dt$. We shall prove that the inner integrals are equal, i.e., for every t, $\int_D f(x,t) dx = \int_{V_t} f(x,t) dx = \int_{U_t} F(y,t) dy$ where $U_t \times \{t\} = U \cap \mathbb{R}^{n-1} \times \{t\}$ (and likewise for V_t). We want to use the induction hypothesis at this stage. The key point is to calculate $\det(D_x k)$. Indeed, $\det(Dh) = \det(D_x k)$ and hence for every fixed t, $x \to k(x,t)$ is a diffeomorphism to its image (why is it $1-1$?). By the induction hypothesis, we are done (why?)

4. Every diffeo can be locally written as a finite composition of primitive diffeos: Firstly, this is true for linear maps $x \to Cx$. Indeed, C is a product/composition of elementary matrices/maps. Note that $R_i \to cR_i + dR_j$ is primitive. What about interchanging rows? This can be done by $(a, b) \to (a+b, b) \to (a+b, b-(a+b) = -a) \to (a+b-a = b, -a) \to (b, a)$. Note that translations can also be written (one coordinate at a time) as a composition of primtive translations. Now given $g : A \to B$ and $p \in A$, by translations and multiplications by invertible matrices (which are all compositions by primitive diffeos anyway), we can assume WLog that $p = 0, g(0) = 0, Dg_0 = I$. Now consider $\alpha(x) = (g_1(x), g_2(x), \ldots, g_{n-1}(x), x_n)$. Then $\det(D\alpha)(0) = 1 \neq 0$ and hence α is a local diffeo. Now consider $h = g \circ \alpha^{-1}$. Note that $h(y) = (y_1, \ldots, y_{n-1}, g_n(\alpha^{-1}(y)))$. Now $g = h \circ \alpha$ and h, α are primitive.

□

3 Change of variables and volumes of manifolds (with or without boundary)

As an application of change of variables, we see that if $S \subset \mathbb{R}^n$ is rectifiable and $h(x) = Ax$ (where A is a matrix), then $v(h(S)) = |\det(A)| v(S)$. Indeed, if A is invertible, then h is a change of variables and hence $v(h(S)) = \int_{h(S)} 1 = \int_S 1 |\det(A)| = v(S)|\det(A)|$. If A is not invertible, then $h(S)$ is contained in a subspace of dimension $< n$ and hence has measure zero.
Def: If $a_1, \ldots, a_k$ are linearly independent vectors in $\mathbb{R}^n$, then the set $x = \sum_i c_i a_i$ where $0 \leq c_i \leq 1$ is called the k-dim parallelopiped formed by $a_1, \ldots, a_k$.
If $k = n$, since $h(x) = Ax$ takes the standard unit square to the parallelopiped, the volume of a parallelopiped is $|\det(A)|$. This $|.|$ makes for an interesting definition: An ordered basis $a_1, \ldots, a_n$ is said to be positively oriented if $\det(A) > 0$ and is negatively oriented if $\det(A) < 0$. Note that if S is a change of basis, then S preserves orientation if $\det(S) > 0$.
Here is another interesting observation: Let O be an orthogonal matrix (note that an orthogonal matrix preserves inner products). Then $v(OS) = v(S)$ because $\det(O) = \pm 1$. This observation leads to a nice definition of the volume of a k-dimensional parallelopiped where $k < n$:

Theorem 2. *There is a unique function $V : \mathbb{R}^n \times \mathbb{R}^n \ldots \to \mathbb{R}_{\geq 0}$ such that*

1. *If O is an orthogonal matrix, then $V(Oa_1, \ldots, Oa_k) = V(a_1, \ldots, a_k)$.*

2. *If $a_1 = (b_1, 0), \ldots, a_k = (b_k, 0) \in \mathbb{R}^k \times \{0\}$, then $V(a_1, \ldots, a_k) = |\det(B)|$.*

V vanishes iff a_i are linearly independent. Moreover, $V(a_1, \ldots, a_k) = \sqrt{\det(A^T A)}$ where A is the $n \times k$ matrix $A = [a_1 \ \ldots]$.