MA 200 - Lecture 28

1 Recap

1. Defined pullbacks and proved properties.
2. Proved Stokes' theorem in $\mathbb{H}^n$.
3. Defined smooth forms (and $d\omega$) on manifolds-with-boundary.

2 Differential forms on manifolds (with or without boundary)

We now define integration of top forms on manifolds-with-boundary. Let ω be a smooth k-form field on an oriented smooth k-manifold-with-boundary M. Assume that ω is compactly supported in a smooth orientation-compatible coordinate parametrisation $\alpha : U \to M$. Then we define $\int_M \omega := \int_U \alpha^*\omega$. This definition is well-defined because suppose $\beta : V \to M$ is another orientation-compatible coordinate parametrisation containing the support of ω, then since $\beta = \alpha \circ \phi$ where ϕ is an orientation-preserving diffeomorphism (why?), we see that $\int_U \alpha^*\omega = \int_V \phi^*(\alpha^*\omega) = \int_V (\alpha \circ \phi)^*\omega = \int_V \beta^*\omega$. This integral is certainly linear in ω.
Suppose ω is a general smooth k-form field on a compact oriented smooth k-manifold-with-boundary M, then cover M by orientation-compatible smooth coordinate parametrisations α_i. Let ρ_i be a partition-of-unity subordinate to the open cover $\cup_i \alpha_i(\tilde{U}_i)$ (where $\tilde{U}_i$ are open subsets of $\mathbb{R}^n$ containing U_i to which α_i extend smoothly). Define $\int_M \omega = \sum_i \int_M \rho_i \omega$.
This definition coincides with the previous one when ω is supported in a coordinate parametrisation: Indeed, $\sum_i \int_M \rho_i\omega = \sum_i \int_U \alpha^*(\rho_i\omega) = \int_U \alpha^*(\sum_i \rho_i\omega) = \int_U \alpha^*\omega$.
Moreover, this definition is independent of the partition-of-unity chosen: Suppose ψ_j is another partition-of-unity. Then $\sum_j \int_M \psi_j\omega = \sum_j \sum_i \int_M \rho_i\psi_j\omega = \sum_i \sum_j \int_M \psi_j\rho_i\omega = \sum_i \int_M \rho_i\omega$.
This integral is also linear in ω.
Of course this definition is painful to work with in practice. However, just as in the case of scalar fields, we can prove that it is enough to cover M-upto-measure-zero by disjoint sets that are images of orientation-compatible coordinate parametrisations and hence evaluate the integrals and add them up (without any partition-of-unity).

3 The generalised Stokes theorem

Theorem 1. *Let $M \subset \mathbb{R}^n$ be a smooth compact oriented k-manifold-with-boundary. Let ω be a smooth $n-1$-form-field on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

if ∂M is endowed with the induced orientation (that is, the restriction if $dim(M)$ is even and opposite otherwise).

Proof. $\int_M (d\omega) = \sum_i \int_M \rho_i d\omega = \sum_i \int_M d(\rho_i \omega) - \int_M \omega \sum_i d(\rho_i) = \sum_i \int_M d(\rho_i \omega)$. Since the RHS is $\sum_i \int_{\partial M} \rho_i \omega$, we can assume WLOG that ω is compactly supported in a coordinate parametrisation. Then $\int_M d\omega = \int_U \alpha^*(d\omega) = \int_U d(\alpha^* \omega)$. If $U \subset \mathbb{R}^n$ is open, then this integral is zero. Moreover, the RHS is trivially zero in this case. If $U \subset \mathbb{H}^n$, then the earlier Stokes theorem shows that this integral is $\int_{x_n=0} (-1)^n \alpha^* \omega$. The induced orientation's definition implies that this integral is $\int_{\partial M} \omega$. □

Using this version of Stokes, we can recover our UM 102 theorems:

1. Green: Let $\Omega \subset \mathbb{R}^2$ be an open set whose topological boundary is a collection of simple closed bounded parametrisated smooth curves that are smooth compact 1-manifolds. Then $\bar{\Omega}$ is a smooth manifold-with-boundary (whose boundary is the topological boundary) - HW. Let P, Q be smooth functions on $\bar{\Omega}$. Then $\int_C (Pdx + Qdy) = \int_\Omega d(Pdx + Qdy) = \int_\Omega (Q_x - P_y) dxdy$ provided C is oriented with the restricted orientation, i.e., in the UM 102 way.

2. Stokes: Let $M \subset \mathbb{R}^3$ be a smooth compact oriented surface-with-boundary. Let $\vec{F}$ be a smooth vector field on M. Then let $\omega = F_1 dx + F_2 dy + F_3 dz$. Stokes implies $\int_M d\omega = \int_{\partial M} \omega$. Now suppose α is an orientation-compatible parametrisation of the interior. Then $\alpha^*(d\omega) = \alpha^*((\nabla \times \vec{F})_1 dy \wedge dz + \ldots) = (\nabla \times \vec{F})_1 \circ \alpha(\partial_u \alpha_2 \partial_v \alpha_3 - \partial_u \alpha_3 \partial_v \alpha_2) + \ldots$ which corresponds to $(\nabla \times \vec{F}).d\vec{A}$.

3. Divergence: Let $\Omega \subset \mathbb{R}^3$ be an open set whose topological boundary is a collection of smooth compact surfaces. Then $\bar{\Omega}$ is a smooth compact 3-manifold-with-boundary whose boundary is the topological boundary) - HW. Now let $\vec{F}$ be a smooth vector field on $\bar{\Omega}$. Consider the 2-form $\omega = F_1 dy \wedge dz + \ldots$. Then $d\omega = \nabla.\vec{F} dx \wedge dy \wedge dz$. Thus the generalised Stokes theorem gives us what we want.