# MA 200 - Lecture 23

## 1 Recap

1. Defined surface areas/volumes of parametrised manifolds-without-boundary.
2. Defined manifolds-with-boundary.

## 2 Manifolds-with-boundary

Def: Let $M \subset \mathbb{R}^n$ be a $k$-dimensional manifold-with-boundary. A point $p \in M$ is said to be *interior* (NOT in the topological sense) if a neighbourhood of it is coordinate parametrised by an open subset of $\mathbb{R}^k$. It is said to be a boundary point otherwise. The set of boundary points (denoted as $\partial M$) is called the boundary of $M$.
The following criteria are useful when $p \in V \subset M$ and $\alpha : U \to V$ is a coordinate parametrisation:

1. If $U \subset \mathbb{R}^k$, $p$ is an interior point (by definition).
2. If $U \subset \mathbb{H}^k$ but $p$ is in $\mathbb{H}^k_{x_k>0}$ then $p$ is an interior point (indeed simply shrink $U$).
3. If $U \subset \mathbb{H}^k$ and $p$ is on $x_k = 0$, then $p$ is a boundary point (indeed, if not, then a neighbourhood of such a point on $\mathbb{H}^k$ is homeomorphic to an open subset of $\mathbb{R}^k$ by means of a $C^r$ map $f$ whose derivative is an isomorphism. But by the IFT, the image of $f^{-1}$ is open in $\mathbb{R}^k$ whereas the original neighbourhood in $\mathbb{H}^k$ isn't).

Finally, if $M$ is a $k$-dimensional manifold-with-boundary such that $\partial M \neq \phi$ (typically, this condition is understood), then $\partial M$ is a $k-1$-dimensional manifold-without-boundary in $\mathbb{R}^n$: Indeed, cover $\partial M$ by open sets $\alpha_i(U_i) \cap M = V_i \cap M$ that are boundary coordinate parametrisations for $M$. Now consider the maps $\tilde{\alpha}_i(x_1, \ldots, x_{k-1}) = \alpha_i(x_1, \ldots, x_{k-1}, 0)$ from $U_i \cap \{x_k = 0\}$ to $V_i \cap \partial M$. These are $C^r$ bijective maps and $D\tilde{\alpha}_i(v_1, \ldots, v_{k-1}) = D\alpha_i(v_1, \ldots, v_{k-1}, 0)$ which is $0$ iff $v_j = 0 \, \forall \, j$. Moreover, $\tilde{\alpha}_i$ are homeomorphisms to their images because $\tilde{\alpha}_i^{-1}$ are restrictions of the continuous functions $\alpha_i^{-1}$. Hence these are coordinate parametrisations.
Lastly, here is a theorem that allows us to prove for instance that the unit disc is a manifold-with-boundary. (Another example is the hemisphere (HW) but it does not follow from this theorem.)

**Theorem 1.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be $C^r$. Let $N = \{x \mid f(x) \leq 0\}$ and let $\nabla f \neq 0$ for every point on $f^{-1}(0) \neq \phi$. Then $N$ is an $n$-dimensional manifold-with-boundary in $\mathbb{R}^n$ and $\partial N = f^{-1}(0)$.*

*Proof.* Note that the set $f < 0$ is open (and hence a manifold-without-boundary) in $\mathbb{R}^n$. Let $p \in f^{-1}(0)$. Assume that $\partial_i f \neq 0$. Consider the map $H : U \to \mathbb{R}^n$ given by $H(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, -f)$. Now $\det(DH(p)) \neq 0$ and hence $H$ is a local diffeo from $V \subset U \to H(V)$ near $p$. Moreover, $H_n \geq 0$ iff $f \leq 0$. Thus, $\alpha = H^{-1}$ takes the open set $H(V) \cap \mathbb{H}^k$ homeomorphically to its image and is a boundary coordinate parametrisation. Moreover, $H_n = 0$ iff $f = 0$. Thus $f^{-1}(0) = \partial M$. Lastly, the topological boundary of $f < 0$ is $f = 0$ (why?) □

We now define the integral of certain kinds of functions over manifolds with or without boundary:
Let $M$ be a compact manifold with nonempty or without boundary in $\mathbb{R}^n$ of dimension $k$. Let $f : M \to \mathbb{R}$ be continuous and compactly supported in $\alpha(U)$ where $\alpha : U \to M$ is a coordinate parametrisation. We define $\int_M f dV = \int_U (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)}$. Note that this *a priori* improper integral is actually an ordinary Riemann integral (because $f \circ \alpha$ has compact support), we can assume $U$ is bounded WLog, and the integral can be taken over $Int(U)$ (because the boundary, being possibly a subset of $\mathbb{H}^n$ has measure zero anyway).
The key point is that this integral is well-defined, independent of the coordinate parametrisation chosen: This follows from the change-of-variables formula and the fact that we are eventually integrating over an open set anyway. Moreover, linearity holds, i.e., $\int (af + bg) = a \int f + b \int g$ by the linearity of the usual integral.
Here is the general definition: Let $M$ be a compact manifold with nonempty or without boundary in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a continuous function. Let $\phi_i$ be a (finite) partition-of-unity subordinate to a cover by *all* coordinate parametrisations. Then $\int_M f dV := \sum_i \int_M \phi_i f dV$. When $f = 1$, $\int_M 1 dV$ is called the surface area/volume of $M$.

1. If $f$ has support in a coordinate patch, this definition coincides with the earlier one: $\sum_i \int_M \phi_i f dV = \sum_i \int_{Int(U)} (\phi_i f) \circ \alpha \sqrt{\det(D\alpha^T D\alpha)} = \int_{Int(U)} \sum_i \phi_i \circ \alpha f \circ \alpha \sqrt{\det(D\alpha^T D\alpha)}$ and we are done.

2. It is independent of the partition-of-unity: If $\psi_j$ is another partition-of-unity subordinate to another cover, then $\sum_j \int_M \psi_j f dV = \sum_j \int_M \sum_i \phi_i(\psi_j f) dV = \sum_j \sum_i \int_M \phi_i \psi_j f dV$ (by linearity) and by linearity again, this equals $\sum_i \int_M \sum_j \phi_i \psi_j f dV = \sum_i \int_M \phi_i f dV$.

Linearity of this general definition is also easy to prove.
Now of course this definition is impossible to work with practically speaking. Thankfully, it is not difficult to prove (using the fact that $C^1$ maps take measure zero sets to measure zero sets) that if you cover-upto-measure zero (measure zero on a manifold simply means that it can be covered by countable many coordinate patches where it has measure zero) a compact manifold (with or without boundary) by finitely many coordinate patches, then the integral is simply the addition of the improper integrals over each coordinate patch. As a consequence, we can calculate the surface area of a sphere using the usual parametrisation (upto measure zero).

## 3 A sketch of proof of Green's theorem

A version of Green: Let $\Omega \subset \mathbb{R}^2$ be a closed set, $f : \Omega \to \mathbb{R}$ a smooth function, $f = 0$ is a regular level set, and $f \geq 0$ is $\Omega$. Suppose $\partial\Omega$ can be parametrised-upto-measure zero by a single patch $\gamma : (0,1) \to \mathbb{R}^2$ such that $\nabla f \times \gamma'$ points in the $\hat{k}$ direction throughout $\gamma(0,1)$. Let $P, Q : \Omega \to \mathbb{R}$ be smooth functions. Then $\int_0^1 ((P \circ \gamma)\gamma_1' + (Q \circ \gamma)\gamma_2')dt = \int_{Int(\Omega)} (Q_x - P_y)$.

*Proof.* Cover the boundary by boundary coordinate patches $U_i$ (of the form $\alpha^{-1}(x, y) \to (-x, -f)$ (when $f_y > 0$) or $(y, -f)$ (when $f_x > 0$) or $(x, -f)$ (when $f_y < 0$) or $(-y, -f)$ (when $f_x < 0$). Note that these changes of variables have positive Jacobians) and the interior by the usual patch $V$. Choose a partition-of-unity $\rho_j$ subordinate to this cover. By linearity, we can assume WLog that $P, Q$ have supports in one of these coordinate patches. If that patch is $V$, then the RHS is zero (because it is trivial to prove Green for rectangles) and so is the LHS. If it is one of the $U_i$, then by change of variables, we can reduce to a rectangle and be done. Now the fact that the integral can be calculated by only one patch $\gamma(0, 1)$ follows from the measure zero business. The key point is that the direction of $\gamma'$ is the right one for the Green theorem over a rectangle. □

## 4 Orientability of manifolds

In the above sketch of proof, it appears crucial that the integral be such that it changes by the Jacobian upon change of variables and that we have successfully covered the manifold-with-boundary by coordinate patches where the change of patch Jacobian is *positive*. We generalise the latter property into a definition as follows. (The former property will also have to be generalised to higher dimensions.)
Let $g : A \subset \mathbb{R}^k \to B \subset \mathbb{R}^k$ be a diffeo. It is said to be orientation-preserving if $\det(Dg) > 0$ everywhere. It is said to be orientation-reversing if $\det(Dg) < 0$ everywhere. (Note that if $A$ is connected, then only one of these possibilities occurs.)
Let $M \subset \mathbb{R}^n$ be a $k$-dimensional manifold with nonempty or without boundary. Given two coordinate parametrisations $\alpha_i : U_i \to V_i$, we say that they are orientation-compatible with each other if the transition functions $\alpha_i \circ \alpha_j^{-1}$ are orientation-preserving. If $k \geq 2$, and $M$ can be covered with coordinate patches that are mutually orientation-compatible with each other, then $M$ is said to be orientable and the given collection of compatible coordinate patches, augmented with all possible coordinate patches that are compatible with the given ones, is said to be an orientation of $M$.
Given a parametrisation $\alpha$, we can *reverse* its orientation: $\beta = (-\alpha_1, \alpha_2, \ldots)$. Now $\alpha \circ \beta^{-1}(x_1, x_2, \ldots, x_k) = (-x_1, \ldots, x_k)$, which is an orientation-reversing diffeo. Thus, given an oriented manifold, we can reverse all the orientation-compatible parametrisations and produce *another* orientation called the opposite orientation.