MA 200 - Lecture 24

1 Recap

1. Defined the boundary ∂M of a manifold-with-boundary.
2. Proved that $f \geq 0$ for certain functions give us manifolds-with-boundary.
3. Defined integrals of functions over manifolds-with-boundary and mentioned that to practically calculate them, it is enough to cover the manifold-upto-sets-of-zero-measure by disjoint coordinate parametrisations and add the integrals up.
4. Defined orientability and orientation of manifolds (motivated from change-of-variables used in the sketch of the proof of Green).
5. Defined tangent spaces (warning: T_pM of a manifold-with-boundary has the same dimension at all points including the boundary points!)

2 Orientability of manifolds

1. An oriented 1-manifold M (according to the $k \geq 2$ definition) has a C^r-varying unit-speed tangent vector field, i.e., a function $T : M \to \mathbb{R}^n$ such that $T(p) \in T_pM \,\forall\, p \in M$, for any parametrisation α, $T \circ \alpha$ is a C^r function, and $\|T(p)\| = 1 \,\forall\, p$. Indeed, cover M with orientation-compatible coordinate parametrisations. Then define $T(p) = \frac{\alpha_i'(t)}{\|\alpha_i'(t)\|}$. This definition is independent of i.
   However, the converse is not true in the case of $k = 1$. The problem is that $[0,1] \subset \mathbb{R}$ is not "orientable": Indeed, if there is such a $T : [0,1] \to \mathbb{R}$ that is compatible with orientation-preserving charts, then suppose the usual interior coordinate chart is orientation-compatible on $\mathbb{R}$. The boundary charts near 1 and 0 "point" in opposite directions and hence we have a problem. Thus we *define* orientation for 1-manifolds to simply be the existence of a C^r-varying unit-speed tangent vector field (the opposite orientation comes from simply $-T$).
2. Here is a lemma that produces several examples: Let M be an $n-1$-dimensional (where $n-1 \geq 2$) manifold in $\mathbb{R}^n$. A C^r-varying unit normal vector field on M is a function $n : M \to \mathbb{R}^n$ such that for any coordinate patch α, $n \circ \alpha$ is C^r (by the chain rule, this can be accomplished by making sure such is the case for *some* collection of patches that cover M), $n(p) \perp T_pM \;\forall\; p$. Now M is orientable iff it has a C^r-varying unit normal vector field:

(a) M is orientable: Consider a cover by orientation-compatible coordinate parametrisations α_i. Then note that at each point, there are only two choices of unit normals (why?) we make the choice by requiring that $\det(\vec{n}(p)\ \frac{\partial \alpha_i}{\partial u_1}(\alpha^{-1}(p))\ \frac{\partial \alpha_i}{\partial u_2}\alpha^{-1}(p)\ \ldots) > 0$. In the HW you will show that such an $\vec{n}$ is C^r by producing an explicit formula for it.

(b) There is a C^r-varying unit normal vector field $\vec{n}$: Consider all parametrisations α such that $\det(\vec{n}(p)\ \frac{\partial \alpha_i}{\partial u_1}(\alpha^{-1}(p))\ \frac{\partial \alpha_i}{\partial u_2}(\alpha^{-1}(p))\ \ldots) > 0$. They form an orientation. Indeed, they exist: If we take any parametrisation near p such that this condition fails at p, simply reverse it (and the condition will be met in a neighbourhood of p). They are compatible: Use the chain rule (HW).

In other words, if we take a surface in $\mathbb{R}^3$ defined by $f = 0$ where $\nabla f \neq 0$ on $f = 0$, then it is orientable. Concretely, we can take say, the sphere S^2, and explicitly produce orientated coordinate parametrisations.

3. A Möbius strip is not orientable (a challenging exercise).

4. Suppose $M \subset \mathbb{R}^3$ is an 3-dimensional manifold-with-boundary given by $f \leq 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is C^r and $\nabla f \neq 0$ on $f = 0$. Then cover $f < 0$ by the usual coordinate chart. Near ∂M, consider $\vec{n} = \frac{\nabla f}{\|\nabla f\|}$. Note that the boundary charts $(y_1, y_2, y_3) = (-x_2, x_3, -f)$ (if $f_1 > 0$), $(x_1, x_3, -f)$ (if $f_2 > 0$), $(-x_1, x_2, -f)$ (if $f_3 > 0$), etc are orientation-compatible with each other and the usual interior chart. Moreover, the restrictions on the boundary, i.e., to $f = 0$ form an orientation for the boundary (why?).

5. More generally, if M is an oriented manifold-with-nonempty-boundary, then ∂M is orientable and inherits a standard god-given orientation (that coincides in the case of domains in $\mathbb{R}^2$ with the orientation needed for Green's theorem): Consider boundary charts $(u_1, \ldots, u_k) \in \mathbb{H}^k \to \alpha(u) \in \mathbb{R}^n$ that are orientation-compatible. Restrict them to the boundary, i.e., $(v_1, \ldots, v_{k-1}) \to \alpha(v_1, \ldots, v_{k-1}, 0)$. Suppose β is another parametrisation such that on the overlap, the transition map $g = \beta^{-1} \circ \alpha$ is orientation preserving on an open subset of $\mathbb{H}^k$. On $\partial \mathbb{H}^k$, the last row of Dg is $Dg_k = (0, 0, 0 \ldots, 0, \frac{\partial g_k}{\partial x_k})$. Since $\det(Dg) > 0$, $\frac{\partial g_k}{\partial x_k} \neq 0$. It is in fact, > 0 because if we move in the direction of x_k, i.e., into the open set, then we will do so in the image too. Thus, $\det(\frac{\partial(g_1, \ldots, g_{k-1})}{\partial(x_1, \ldots, x_{k-1})}) > 0$ and this is precisely the Jacobian of the restriction of the coordinate parametrisations. We say that ∂M has the *induced* orientation from M if n is even and we equip ∂M with the orientation given by the restrictions of these charts. If n is odd, we choose the opposite orientation. This god-given orientation *coincides* (HW) with the following: Parametrisations $\gamma(u_1, \ldots, u_{k-1})$ of the boundary such that at every point $\det(\vec{n}(p)\ \frac{\partial \gamma}{\partial u_1}(p)\ \frac{\partial \gamma}{\partial u_2}(p) \ldots) > 0$ where $n(p)$ an the outward-pointing vector, i.e., take any boundary parametrisation α, then $\vec{n}(p) = -\frac{\partial \alpha}{\partial x_n}(\alpha^{-1}(p))$.
So for instance, in Green's theorem, moving "anticlockwise" along the "outer boundary", i.e., moving along the boundary (with your head held high because you are doing maths) such that the region lies to your left, means that $outward\ normal\ \times\ tangent\ vector\ = \hat{k}$ which is the correct orientation.

3 Differential forms, wedge products, and form-fields in $\mathbb{R}^n$

We want to generalise the notion of a cross product (because that will also help us generalise the notion of curl). Naively, if $a, b \in \mathbb{R}^4$, then $a \times b$ ought to have components $a_i b_j - a_j b_i$ where $i \neq j$. Even if we demand $i < j$ (because anyway, $a_i b_j - a_j b_i = -(a_j b_i - a_i b_j)$, i.e., $a \times b = -b \times a$), the number of components is 6. So $a \times b$ cannot be a vector in $\mathbb{R}^4$! (On the other hand, $a \times b \times c$ in this naive prescription will have only 4 components!) So we need to extend our definitions from vectors to other beasts. Whatever this naive $a \times b$ is, each component is certainly multilinear in a, b and antisymmetric. That motivates the following definitions.

Let V be a f.d real vector space. We will almost always consider $V = \mathbb{R}^k$ or $T_p M$ for some manifold-with-boundary. We know what a multilinear map $T : V \times V \times \ldots \to \mathbb{R}$ is (what is it?) Multilinear maps from V^m to $\mathbb{R}$ are also called m-tensors. An example is $T(\vec{v}, \vec{w}) = \det(\vec{v}\ \vec{w})$ where $\vec{v}, \vec{w} \in \mathbb{R}^2$. Another is $p(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2 = \langle v, w \rangle$. The set of m-tensors forms a vector space. We can find a nice basis for this vector space: Given an m-multiindex $I = (i_1, \ldots, i_m)$ and a basis $e_1, \ldots, e_n$, the tensors $\phi_I(v_1, \ldots, v_m) = (v_1)_{i_1} (v_2)_{i_2} \ldots$ form a basis (so the dimension is n^m where $n = dim(V)$). By the way, for a single index, the 1-tensors ϕ_i are called the dual basis of e_j. 1-tensors are also called linear functionals, or as we shall call them later, 1-forms. (By the 0-forms are simply real numbers.)

A neat way of phrasing this statement is through the definition of the tensor product of two tensors: Suppose S, T are k and l tensors respectively. The map $S \otimes T$ defined as $S \otimes T(v_1, \ldots, v_k, w_1, \ldots, w_l) = S(v)T(w)$ is a $(k+l)$-tensor called the tensor product of S and T. Note that $\phi_I = \phi_{i_1} \otimes \phi_{i_2} \ldots$. This tensor product obeys some standard properties (that I believe Apoorva had discussed)- associativity and linearity.

Given a linear map $T : V \to W$, there is a 'dual' linear map denoted as T^* from k-tensors on W to those on V: $T^* L(v_1, \ldots, v_k) = L(Tv_1, \ldots, Tv_k)$. This dual map obeys some (easy to verify) properties: T^* is linear on the space of k-tensors on W, $T^*(f \otimes g) = T^* f \otimes T^* g$ and $(S \circ T)^* = T^* \circ S^*$.

What we want to study are tensors that are *antisymmetric* or *alternating* (akin to our example of determinants, or cross products):

Def: A k-tensor is said to be *symmetric* if $f(v_1, \ldots, v_k) = f(v_1, \ldots, v_{i+1}, v_i, \ldots)$, i.e., interchanging adjacent arguments doesn't change the value. By a finite number of *adjacent* interchanges, one can see that interchanging *any* two arguments keeps the value invariant. Since every permutation $\sigma \in S_k$ is a product of transpositions (hopefully you did this in UM 205), the value is invariant under any permutation of the arguments. A k-tensor is said to be *antisymmetric* or *alternating* $f(v_1, \ldots, v_k) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots)$. Such tensors are also called k-forms and they form a vector space that is denoted as $\Lambda^k V$. (For $k = 1$, every tensor is trivially alternating.) They are also called k-forms. Clearly, if $v_i = v_j$ for some $i \neq j$, the value of f is zero on such a tuple (if $|j - i| = 1$, this is trivial. Now induct on $|j - i|$). Moreover, if a tensor is such that it is zero whenever adjacent arguments coincide, then it is alternating: $0 = f(v_1, \ldots, v_i + v_{i+1}, v_i + v_{i+1}, v_{i+2}, \ldots)$. Now use multi-linearity and the vanishing property again to be done.

Inner products are examples of symmetric 2-tensors and the determinant an example of $\Lambda^n(\mathbb{R}^n)$.