MA 200 - Lecture 16

1 Recap

1. Taylor's theorem proof.
2. Reduced the proof of the second derivative test to a linear-algebraic lemma.

2 Taylor's theorem and the second derivative test

Lemma 2.1. *A real symmetric matrix is positive-definite if and only if all of its eigenvalues are positive. (It is positive-semidefinite iff all of its eigenvalues are non-negative.) Moreover, if $H : U \subset \mathbb{R}^n \to Sym_{n\times n}(\mathbb{R})$ is a continuous function and if $H(a)$ is positive-definite, then $H(\theta)$ is positive-definite for all $\theta \in B_a(\epsilon)$ for some $\epsilon > 0$.*

Proof. If A is symmetric, then by the spectral theorem, there exists an orthogonal matrix O such that $O^T A O = D$ where D is the diagonal matrix consisting of eigenvalues of A. As a consequence, if A is positive-definite (semidefinite), then $v^T A v > 0$ ($v^T A v \geq 0$) for all $v \neq 0$. Hence $(Ov)^T A(Ov) > 0$ (non-negative) for all $v \neq 0$. Thus, $v^T D v > 0$ (non-negative) and thus the eigenvalues are positive. Tracing the argument backwards (indeed, O is invertible), we see that A is positive-definite if its eigenvalues are positive. Now assume $H(a)$ is positive-definite. Suppose no ϵ works, i.e., for every n there is a $\theta_n \in B_a(1/n)$ such that $v_n^T H(\theta_n) v_n \leq 0$ (and $\|v_n\| = 1$). Then since the unit sphere is compact, there exists a convergent subsequence v_{n_k} converging to v on the unit sphere. Moreover, $\theta_{n_k} \to a$ (why?) By continuity of the function H, $v^T H(a) v \leq 0$ but $v \neq 0$. Therefore we have a contradiction. □

When does the Taylor series converge? For instance, $1 + x + x^2 + \ldots$ is the formal Taylor series of $\frac{1}{1-x}$ around $x = 0$. It certainly does not converge if $x > 1$ for instance. When $-1 < x < 1$, it does converge to $\frac{1}{1-x}$ (why?)
Here is a supremely strange example:
Consider $f(x) = e^{-1/x^2}$ when $x > 0$ and $f(x) = 0$ when $x < 0$. This function is C^∞ everywhere: Indeed, it is so away from $x = 0$ (compositions of smooth functions are smooth). The claim is that it is smooth at the origin too (with all derivatives equal to 0). Indeed, we claim that there exists a polynomial $p_k(t)$ of degree $3k$ such that $f^{(k)}(x) = p_k(1/x)e^{-1/x^2}$ when $x > 0$ and 0 when $x \leq 0$: For $k = 0$ this is true. Assume truth for $0, 1, 2, \ldots, k-1$. Then $f^{(k-1)}(x) = p_{k-1}(1/x)e^{-1/x^2}$ when $x > 0$ and 0 when $x \leq 0$. So $f^{(k)}(0) = \lim_{h\to 0} \frac{p_{k-1}(1/h)e^{-1/h^2}}{h} = 0$ (by the squeeze rule and the fact that $g(x)e^{-x}$ goes to 0 when g is a polynomial and $x \to \infty$). When $x > 0$,

$f^{(k)}(x) = \left(\frac{-1}{x^2}p'_{k-1}(1/x) + \frac{2}{x^3}p_{k-1}(x)\right)e^{-1/x^2}$. □

This means that the Taylor series converges (it is identically zero!) but NOT to the original function! (By the way, a variant of this function plays a role in physics: Look up the KT-phase transition).

This function leads to a very interesting phenomenon: By reflecting, i.e., $g(x) = f(-x)$, and translating, i.e., $h(x) = g(x-a)$ (where $a > 0$), and multiplying, i.e., $k(x) = h(x)f(x)$, we get a smooth function that is *identically* zero outside a compact set! This leads to a definition:

Def: Let $f : X \to \mathbb{R}^n$ be a continuous function and X be a metric space. Then the closure of the set $\{x \in X | f(x) \neq 0\}$ is called the *support* of f. (It is a closed set by definition.) f is said to have compact support if its support is compact. (Note that if X is itself compact, every continuous function has compact support.)

In other words, we have found a smooth function on $\mathbb{R}$ having compact support! (It is easy to come up with continuous compactly supported functions.) Not just that, we can have some more fun: Note that by further translation (and scaling if necessary), we can easily make sure that the support is *any* compact interval of our choice!

Now, upon integration, i.e., $l(x) = \int_{-\infty}^{x} k(t)dt$, we obtain a smooth function that is a constant on $x \leq 0$ and $x \geq a$. The same tricks as before allow us to produce a function that has compact support on a compact interval of our choice and is identically 1 on a sub-interval of our choice!

3 Integration in more than one-variable

Let $Q \subset \mathbb{R}^n$ be a closed rectangle $[a_1, b_1] \times [a_2, b_2] \ldots [a_n, b_n]$. The volume of Q is defined to be $v(Q) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$. The "width" is the maximum of $b_i - a_i$ and the intervals $[a_i, b_i]$ are called the component intervals of Q.

Let $f : Q \to \mathbb{R}$ be a bounded function. We want to define $\int_Q f dV$. To this end, roughly speaking, we split Q into sub-rectangles where f is roughly a constant, and add up the resulting numbers. We are led to some definitions:

Def: Given $[a, b] \subset \mathbb{R}$, a partition P is a set of points $a = t_0 < t_1 < \ldots < t_k = b$. The sub-intervals of the partition are $[t_i, t_{i+1}]$. Given a rectangle Q, a partition of Q is the set $P_1 \times P_2 \ldots P_n$ where P_i are partitions of $[a_i, b_i]$. The Cartesian products of the sub-intervals yield several subrectangles of the partition. The maximum width of all these subrectangles is called the mesh of the partition (so the smaller the mesh, the more the number of sub-rectangles we are dividing into). A partition P' is said to be finer than a partition P (or P' is said to be a refinement of P) if $P'_i \subset P_i$ for every i. Given any two partitions $P = P_1 \times P_2 \times \ldots$ and $P' = P'_1 \times \ldots$, the partition $C = (P_1 \cup P'_1) \times (P_2 \cup P'_2) \ldots$ is finer than P and P' and is called their common refinement.

Def: Let P be a partition of Q. For every subrectangle R, let $m_R(f)$ be the infimum of f and $M_R(f)$ be the supremum of f over R. The lower Riemann sum $L(P, f) = \sum_R m_R(f)v(R)$ and the upper Riemann sum $U(P, f) = \sum_R M_R v(R)$.

The key point is

Lemma 3.1. *Let P be a partition of a rectangle Q and $f : Q \to \mathbb{R}$ be a bounded function. If P' is a refinement of P, then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.*

Proof. Let k be the number of points in $P_1' - P_1$ plus those in $P_2' - P_2$ plus so on. We induct on k. In fact, we claim that $k = 1$ is enough. Indeed, if it is true for $1, 2 \ldots, k-1$, then replace P with the partition obtained by adding the $k-1$ points and then apply the $k = 1$ case.
For $k = 1$: Suppose the additional point b is in the i^{th} component interval and in the sub-interval $[t_{ij}, t_{ij+1}]$. Then the rectangles $R = I_1 \ldots I_{i-1} \times [t_{ij}, t_{ij+1}] \times I_{i+1} \ldots$ are replaced by $R_1 = I_1 \ldots I_{i-1} \times [t_{ij}, a] \times I_{i+1} \ldots$ union $R_2 = I_1 \ldots I_{i-1} \times [a, t_{ij+1}] \times I_{i+1} \ldots$. The infimum increases if the size of the size is reduced (why?) and the supremum decreases. Hence $m_{R_1}, m_{R_2} \geq m_R$, $M_{R_1}, M_{R_2} \leq M_R$ and since $v(R) = v(R_1) + v(R_2)$, $m_R v(R) \leq m_{R_1} v(R_1) + m_{R_2} v(R_2)$ and likewise for M_R. Thus we are done. $\square$

Moreover, if P, P' are any two partitions, then $L(f, P) \leq U(f, P')$ (and as a consequence, the lower R.I is $\leq$ the upper one): Indeed, let C be their common refinement. Then $L(f, P) \leq L(f, C) \leq U(f, C) \leq U(f, P')$.
Def: Let $f : Q \to \mathbb{R}$ be a bounded function and $Q \subset \mathbb{R}^n$ be a closed rectangle. Then the lower Riemann integral $\underline{\int_Q} f dV$ is the supremum over all partitions of $L(P, f)$ and the upper Riemann integral $\overline{\overline{\int_Q}} f dV$ is the infimum over all partitions of $U(P, f)$. These numbers always exist. f is said to be Riemann integrable with integral $\int_Q f dV$ if these numbers are equal and $\int_Q f dV = \underline{\int_Q} f dV = \overline{\int_Q} f dV$.
Example: A constant function is Riemann integrable with integral $cv(Q)$. Indeed, consider the trivial partition to conclude that the upper and lower Riemann sums and hence the integrals are equal and that too to $cv(Q)$. Now if P is any other partition, since $m_R = M_R = c$, we see that $v(Q) = \sum_R v(R)$ (an interesting identity).
Example: A piecewise-constant function on Q is a partition P_0 together with constants $c_{i_1 i_2 \ldots i_n}$ for each open subrectangle and arbitrary values on the boundaries. Piecewise-constant functions are R.I with integral $\sum_I c_I v(R_I)$ (where I is a multi-index). Indeed, given any partition P, consider the common refinement of P_0, P. Consider an even further refinement by adding points on both sides of the points in $(P_0)_i$ with distance $\epsilon > 0$. Now the lower and upper Riemann sums are within $C\epsilon$ of each other and $\sum_I c_I v(R_I)$ (why?) Hence, the upper and lower R.I are within $C\epsilon$ of each other. Since ϵ is arbitrary, we are done.
Non-example: The Dirichlet function $f(x) = 1$ when x is a rational and $f(x) = 0$ when x is irrational is not R.I over $[0, 1]$.

Theorem 1. *Riemann's criterion: A bounded function $f : Q \to \mathbb{R}$ is R.I iff for every $\epsilon > 0$, there is a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.*

The proof is exactly the same as in the $1-D$ case.