MA 235 - Lecture 16

1 Recap

1. Regular value theorem
2. Sard's theorem
3. Smooth vector fields

2 Vector fields and the tangent bundle

Examples/Non-examples of smooth vector fields:

- Any collection of n smooth functions $X^i : \mathbb{R}^n \to \mathbb{R}$ gives a smooth vector field $X = X^i e_i$ on $\mathbb{R}^n$.
- Cover S^1 with two coordinate charts given by the angles $0 < \theta < 2\pi$, $0 < \psi < 2\pi$ (anticlockwise made with the positive x-axis and negative x-axis). Then on the intersection, when $\pi \leq \theta < 2\pi$, $\psi = \theta - \pi$. When $0 < \theta < \pi$, $\psi = \theta + \pi$. Thus $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \psi}$ on the intersection. Therefore, the vector field given by $\frac{\partial}{\partial \theta}$ on the θ-chart and $\frac{\partial}{\partial \psi}$ on the ψ-chart is a well-defined smooth vector field. (In fact, this vector field spans T_pM at every point.) Likewise, we can come up with n-smooth vectorfields on the n-torus that span T_pM at every point.
- The hairy ball theorem implies that there is no smooth vector field on S^2 that is *nowhere* vanishing. In particular, we cannot come up with two smooth vector fields on S^2 that span T_pS^2 at every point.
- The vector field $X = 0$ is of course a smooth vector field on any manifold (with or without boundary). This is a trivial example.
- Actually, on every manifold (with or without boundary), we can come up with smooth non-trivial vector fields. Indeed, take any coordinate unit ball (B, x) (around an interior point). Take a bump function $\rho : M \to \mathbb{R}$ such that $supp(\rho) \subset B$ and $\rho = 1$ on the ball of radius half. Then $\rho \frac{\partial}{\partial x^1}$ is an example of a smooth vector field on M by extending it to be 0 outside B.

Let M be a smooth manifold (without boundary). Can TM be given the structure of a smooth manifold such that a smooth vector field X is an example of a smooth map from M to TM? We can. The resulting smooth manifold TM is called the *tangent bundle* of M.
Consider the projection map $\pi : TM \to M$ given by $\pi(p, v) = p$. Indeed, cover M with a countable collection of coordinate charts (U_α, x^i_α). For each such chart consider $\pi^{-1}(U_\alpha) = \cup_{p\in U_\alpha} T_pM = \cup_{p\in U_\alpha} T_pU_\alpha = \cup_{p\in\phi_\alpha(U_\alpha)} T_p\mathbb{R}^n$. This set is set-theoretically bijective to $U_\alpha \times \mathbb{R}^n$ by $T_\alpha(x, v) = (x, v^i \frac{\partial}{\partial x^i})$. We now take a countable basis $\mathcal{B}_\alpha$ of $U_\alpha \times \mathbb{R}^n$ and declare the collection $T_\alpha(\mathcal{B}_\alpha)$ over all α as a countable basis for a topology for T_pM. (So in particular, a smooth vector field is a continuous function from M to TM (why?).) This topology is not necessarily homeomorphic to $M \times \mathbb{R}^n$. (Only locally, this sort of a statement is true.)
TM is Hausdorff with this topology (why?) Note tht the map $T^{-1}_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ is a homeomorphism to its image (why?) Thus TM is a topological manifold of dimension $2n$ with these charts. In fact, these charts are smoothly compatible: On $\pi^{-1}(U_\alpha \cap U_\beta)$, the transition map is $(x_\alpha, \vec{v}_\alpha) \to (x_\beta, v^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha} v^j_\alpha)$ which is smooth. Its inverse is also smooth. Thus, considering the smooth structure induced by this countable basis of smooth charts, we can make TM into a smooth manifold.
A smooth vector field is a vector field that is also a smooth map from M to TM (why?)
(HW) If M is a smooth manifold-with-boundary, TM can be made into a smooth manifold-with-boundary such that smooth vector fields are vector fields that define smooth maps from M to TM.

3 Vector bundles

The tangent bundle TM is a special kind of a manifold. It parametrises "a family of smoothly varying tangent spaces". A natural question: Can we parametrise a family of smoothly varying duals of tangent spaces? If $M \subset \mathbb{R}^N$ is a submanifold (or even more generally, $M \subset N$), then another natural question is "Can we parametrise the family of vector spaces that are orthogonal to the tangent spaces"? (We might need such a notion because it might help us answer the question of when $M \subset \mathbb{R}^N$ is a regular level set.)
More generally, what does it mean to have a family of smoothly varying vector spaces?

A trivial family parametrised by M is simply $M \times V$. But even in the case of TM, the most natural topology on TM did *not* make it homeomorphic to $M \times \mathbb{R}^n$. The key point conveyed by TM is the notion of 'local triviality', i.e., it is something that in a neighbourhood secretly looks like/isomorphic to a trivial family of vector spaces. If there is no link between the vector spaces at all, then in what sense are they "smoothly varying"? Indeed in the case of TM, locally, we had smoothly varying bases for T_pM. Using this local smoothly varying basis, we could find a nice bijection between $\pi^{-1}(U_\alpha)$ and $U_\alpha \times \mathbb{R}^n$.
So whatever a "vector bundle" is, it better be a family of vector spaces that is locally trivial, i.e., there must be local smoothly varying bases.
Def: Let M be a smooth manifold (without boundary). A smooth manifold V is said

to be a smooth real vector bundle of rank r over M if there is a surjective smooth map $\pi : V \to M$, $\pi^{-1}(p) = V_p$ is a real vector space of dimension r, and local triviality holds, i.e., for every point $p \in M$, there is a neighbourhood U such that there is a smooth diffeomorphism $T : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that it commutes with the projection maps and $T : V_q \to \mathbb{R}^r$ is a linear isomorphism. Likewise, one can define a smooth complex vector bundle (and continuous real/complex vector bundles over topological manifolds too). The V_p's are called "fibres". Set theoretically, $V = \cup_{p\in M} V_p$. The map T is called a local trivialisation.
The part about local triviality implies the following: There is a smooth collection of functions $s_1, \ldots, s_r : U \to V$ such that $s_i(q) \in V_q \; \forall \; q \in U$ and $s_1(q), \ldots, s_r(q)$ forms a basis for V_q (there is a local smoothly varying basis). Indeed, if local triviality holds, simply take $s_i = T^{-1}(e_i)$.
This motivates the following definition: Let V be a smooth vector bundle over a smooth manifold M. A smooth map $s : M \to V$ such that $s(p) \in V_p$, i.e., $\pi(s(p)) = p$ is called a smooth section. Smooth vector fields are smooth sections of TM.
What we said above is that a local trivialisation gives a collection of smooth sections s_i such that $s_i(p)$ is a basis of V_p. Conversely, given such a collection of smooth sections, the map $L : U \times \mathbb{R}^r \to V$ given by $L(p, \vec{v}) = v^i s_i(p)$ is a smooth $1-1$ map such that L^{-1} is a local trivialisation (HW).

Examples/Non-examples:

- A trivial bundle $M \times \mathbb{R}^r$.
- TM is a vector bundle of rank n over M.
- Möbius bundle: Consider $L = [0, 1]\times\mathbb{R}/\{(0, v) \sim (1, -v)\}$ and $M = [0, 1]/\{0 \sim 1\}$. Note that $f : M \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is a homeomorphism. Defining it to be a diffeomorphism makes M into a smooth manifold diffeo to S^1. It turns out that (proof omitted) L can be made into a smooth vector bundle of rank-1 (such things are called line bundles) over $M = S^1$. One can prove (using the intermediate value theorem) that any smooth section of L over M must vanish somewhere.
- S^2 cannot be a vector bundle over any manifold (why?)