

HW 1 (Quiz in week 2)

1. Suppose $z_{n-i} = (z_{n-i})_0 e^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} z_{n-i+1}(s) ds$ and $z_n(t) = e^{\lambda t} z_n(0)$. Then prove that \vec{z} is a complex linear combination of n complex linearly independent vector-valued functions, each of which solves $\vec{z}' = J\vec{z}$. Now conclude that indeed the space of real solutions of $\vec{y}' = A\vec{y}$ is n real dimensional. Also prove that solutions are unique.
2. (Problem 14 in Chapter 3 in Nandakumaran's book):
 - (a) Consider $y' + py = q$ where p, q are continuous functions. Show that if $q \geq 0$, then $y \geq 0$ if $y(0) \geq 0$.
 - (b) Consider $x' + p_1 x = q$ and $y' + p_2 y = q$. Show that if $p_2 \geq p_1$, $x(0) \geq y(0)$ and $y \geq 0$, then $x \geq y$.
 - (c) Consider $y' + py \leq q$. Derive the inequality

$$y(t) \leq \exp\left(-\int_0^t p(s) ds\right) \left[y(0) + \int_0^t q(s) \exp\left(\int_0^s p(z) dz\right) \right].$$

- (d) Derive Gronwall's inequality: Assume that f, g are continuous on $[a, b]$ and $g \geq 0$. Also assume that $f(t) \leq c + k \int_{t_0}^t f(s)g(s) ds$. Then prove that $f(t) \leq c \exp\left(k \int_{t_0}^t g(s) ds\right)$.

3. Solve the following equation (using the Jordan canonical form): $y' = Ay$ where

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$