

HW 2 (Quiz in week 3)

1. If A, B are matrix-valued differentiable functions such that AB makes sense as a matrix, prove that AB is differentiable and that $(AB)' = A'B + AB'$.
2. Assume existence of a solution to $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = 0$ on $(-\epsilon, \epsilon)$ with $y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}$ (where p_i are continuous functions). Prove that the set of real solutions is an n -dimensional subspace of the space of all n -times differentiable functions.
3. (Slightly tricky) For the previous n^{th} order differential equation, let u_1, \dots, u_n be n solutions. Define the Wronskian W of these solutions. By differentiating it, prove that if W vanishes somewhere, it vanishes everywhere. Now prove that W vanishes somewhere (and hence everywhere) iff u_1, \dots, u_n are linearly independent (note that this property is not true in general but for solutions of ODE, it is so!)
4. Let A be a complex square matrix such that for all eigenvalues of A , their real parts are strictly negative. Then prove that there exist strictly positive constants k, r such that $\|e^{At}\| \leq ke^{-rt} \forall t \geq 0$.