HW 5

1. Consider the ODE $y' = f(y(t), t)$ on $[a, b]$ with $y(a)$ given. Assume that f is smooth on $\mathbb{R} \times \mathbb{R}$, and satisfies $|\frac{\partial f}{\partial y}| \leq L$ on $\mathbb{R} \times \mathbb{R}$.

   (a) Prove that there is a unique solution on $[a, b]$. (We already know there is a solution for some short period of time thanks to existence theorems. Uniqueness is also guaranteed by them. I am asking you to prove that the solution actually exists on $[a, b]$ where b is any given number.)

   (b) Suppose we know that $|y''(t)| + |y'''(t)| \leq M$ for all $t \in [a, b]$. (This is not an unreasonable hypothesis if we know about the derivatives of f.) Then consider the midpoint method $y_0 = y(t_0)$, $y_n = y_{n-1} + hf\left(y_{n-1} + \frac{h}{2} f(y_{n-1}, t_{n-1}), t_{n-1} + \frac{h}{2}\right)$ where $t_n = t_{n-1} + h$, $t_0 = a$, $t_N = b$. Prove that

   $$\left| y(t_n) - y(t_{n-1}) - hf\left(y(t_{n-1}) + f(y(t_{n-1}), t_{n-1})\frac{h}{2}, t_{n-1} + \frac{h}{2} \right) \right| \leq C_1 h^2$$

   for some constant C depending only on L, M.

   (c) Assume that $h < 1$. Prove that $|y(t_n) - y_n| \leq C_2 h^3 + (1 + C_3 h)|y(t_{n-1}) - y_{n-1}|$ for some constants C_2, C_3 depending only on L, M.

   (d) Conclude that $|y(t_n) - y_n| \leq C_4 h^2$ for some constant C_4 depending only on $L, M, t_1 - t_0$.

2. Let u, v be $C^2[a, b]$ functions satisfying

   $$\begin{aligned} (P_1 u')' - Q_1 u &= 0 \\ (P_2 v')' - Q_2 v &= 0, \end{aligned} \tag{1}$$

   where $P_1 \geq P_2 > 0$ are $C^1[a, b]$ functions and $Q_1 \geq Q_2$ are continuous functions. If v does not vanish at any point in $[a, b]$, show that

   $$\left[\frac{u}{v}(P_1 u' v - P_2 u v') \right]_a^b = \int_a^b (Q_1 - Q_2) u^2 dt + \int_a^b (P_1 - P_2)(u')^2 dt + \int_a^b P_2 \frac{(u'v - uv')^2}{v^2} dt. \tag{2}$$

   This formula is known as the Picone formula. Prove that either u is identically zero or u has at most one zero in $[a, b]$. (This is a special case of the Sturm comparison theorem.)