1 Recap

1. Motivated the case for regular singular points (Bessel's equation $y'' + y'/t + y(1 - \nu^2 s/t^2) = 0$ and Legendre's equation).

2 Real-analytic functions

$$a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0, \tag{1}$$

where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \; \forall \; n \geq 0$. For $n = 0$ we see that $f(m) = 0$. This equation is called the indicial equation. It can have real distinct roots, complex distinct roots, or a real (or complex) repeated root. Suppose it has two complex roots m_1, m_2. Then there are two possibilities:

1. $m_1 - m_2$ is not an integer: $f(m+n) \neq 0 \, \forall \, n > 0$. In fact, $|f(m+n)| \geq Cn^2 \, \forall \, n \geq 1$.

2. $m_1 = m_2 + n$ where n is a non-negative integer: In this case $f(m+n) = 0$ at some point and hence we cannot determine all the coefficients for $m = m_2$. So we get only one formal solution y_1 and must try for a different solution using $t^{m_2} \sum b_n t^n + C \ln(t) y_1$.

It turns out (Frobenius' theorem) that $\sum a_n t^n$ and $\sum b_n t^n$ converge on $(-R, R)$ if $\sum p_n t^n$ and $\sum q_n t^n$ do. Here is the proof: First fix $r < R$. We see that $|p_k| + |q_k| \leq \frac{C}{r^k}$.

1. $f(m+n) \neq 0$ (In particular, if $m_1 - m_2$ is not a non-negative integer):

$$|a_n| \leq \frac{C}{n^2} \sum_{k=0}^{n-1} |a_k| \frac{1}{r^{n-k}} \left((m+k) + 1\right)$$
$$\Rightarrow |a_n| \leq \frac{C}{n} \sum_{k=0}^{n-1} |a_k| \frac{1}{r^{n-k}}. \tag{2}$$

   Again, we define $u_n = |a_n|$ with $u_0 = |a_0|$ satisfying

$$u_n = \frac{C}{n} \sum_{k=0}^{n-1} \frac{u_k}{r^{n-k}}. \tag{3}$$

   We see that $u_n \geq |a_n|$ by induction. Thus $u_n = u_{n-1}\left((n-1)/nr + C/n\right)$. Thus $\lim \frac{u_n}{u_{n-1}} = \frac{1}{r}$. Thus $\sum u_n (r-\epsilon)^n$ converges and by the Weierstrass-M test, $\sum a_n t^n$ converges uniformly on $[r-\epsilon, r+\epsilon]$ and since r, ϵ are arbitrary, we are done. We claim that if $m_1 - m_2$ is not an integer, then these two solutions are linearly independent (HW).

2. $m_1 = m_2 + n_0$ where n_0 is a non-negative integer: The above argument works for m_1. Now we try $y_2 = t^{m_2} \sum b_n t^n + C\ln(t)y_1$. Substituting into the ODE we get (after simplification using the fact that y_1 is a solution)

$$(t^{m_2} \sum b_n t^n)'' + P(t)(t^{m_2} \sum b_n t^n)' + Q(t)(t^{m_2} \sum b_n t^n) - \frac{Cy_1}{t^2} + \frac{2Cy_1'}{t} + \frac{CPy_1}{t} = 0. \tag{4}$$