1 Recap

1. Proved Frobenius in the case where $f(m+n) \neq 0$.

2 Real-analytic functions

$$a_n f(m+n) + \sum_{k=0}^{n-1} a_k((m+k)p_{n-k} + q_{n-k}) = 0, \tag{1}$$

where $f(m+n) = (m+n)(m+n-1) + (m+n)p_0 + q_0 \; \forall \, n \geq 0$. For $n = 0$ we see that $f(m) = 0$.

1. $m_1 = m_2 + n_0$ where $n_0 \geq 1$: The above argument works for m_1. For b_n, we see that

$$(t^{m_2} \sum b_n t^n)'' + P(t)(t^{m_2} \sum b_n t^n)' + Q(t)(t^{m_2} \sum b_n t^n) - \frac{Cy_1}{t^2} + \frac{2Cy_1'}{t} + \frac{CPy_1}{t} = 0. \tag{2}$$

Thus defining $a_k = 0 = b_k$ for negative k,

$$C\left((2(m_1+n-n_0)-1)a_{n-n_0} + \sum_{k=0}^{n-n_0} p_k a_{n-n_0-k}\right)$$
$$+ b_n f(m_2+n) + \sum_{k=0}^{n-1} b_k((m_2+k)p_{n-k} + q_{n-k}) = 0. \tag{3}$$

For $n = n_0$ we see that

$$C(2m_1 - 1 + p_0)a_0 + \sum_{k=0}^{n_0-1} b_k((m_2+k)p_{n_0-k} + q_{n_0-k}) = 0$$

and hence we can solve for C (because inductively, we can solve for b_k up to $n_0 - 1$). (By the way, b_{n_0} being a free variable does not give us any more freedom because $b_{n_0} t^{n_0} t^{m_2}$ can be absorbed into $a_0 t^{m_1}$ and hence $b_{n_0} = 0$ Wlog.) Now $f(m+n) \neq 0$ for $n > n_0$. Thus we can solve for all the other b_k. We now have to prove that $\sum_n b_n t^n$ converges absolutely and uniformly on $[-(r-\epsilon), r+\epsilon]$ where $r < R$ and $\epsilon > 0$ are arbitrary. Note that $|a_k| \leq \frac{C}{r^k}$ for all $k \geq 0$. Hence for all sufficiently large n ($\geq N$) we see that

$$|b_n| \leq \frac{C}{n^2} \sum_{k=0}^{n-1} \frac{|b_k|(m_2+k+1)}{r^{n-k}} + \frac{Cn}{r^n}. \tag{4}$$

As usual, we define $u_N = |b_N| > 0$ (without loss of generality) and u_n to satisfy

$$u_n = \frac{C}{n^2} \sum_{k=0}^{n-1} \frac{u_k(m_2+k+1)}{r^{n-k}} + \frac{Cn}{r^n}. \tag{5}$$

Note that $u_n \geq |b_n| > 0$ inductively and that $u_n \geq \frac{Cn}{r^n}\ \forall\ n \geq N+1$. Now we see that

$$\begin{aligned} u_n &= u_{n-1}\left(\frac{n-1}{nr}+\frac{C}{n}\right)+\frac{Cn}{r^n}-\frac{C(n-1)}{r^{n-1}}\left(\frac{n-1}{nr}+\frac{C}{n}\right) \\ \Rightarrow \frac{u_n}{u_{n-1}} &= \left(\frac{n-1}{nr}+\frac{C}{n}\right)+\frac{C}{r^n u_{n-1}}\left(2+\frac{1}{n}-\frac{C(n-1)r}{n}\right) \\ &\Rightarrow \lim \frac{u_n}{u_{n-1}} = \frac{1}{r}. \end{aligned} \tag{6}$$

Hence we are done. □

2. $m_1 = m_2$: In this case we try $y_2 = \ln(t)y_1 + t^m \sum_{n=1}^{\infty} b_n t^n$. The calculations are similar and left as an exercise.