1 Recap

1. Proved Frobenius.

2 General existence and uniqueness theory

We have seen several examples of things that can go wrong in ODE (two solutions (in fact infinitely many (why?)) and a solution that blows up in finite-time). Here are examples where you have no solutions at all

1. $ty' = y$ with $y(0) = 2$. Note that the right-hand-side is 2 at 0 and the left-hand-side is 0. But you may object that I am cheating because it has a singularity.

2. $y' = -1$ when $y \geq 0$ and $y' = 1$ when $y < 0$. The right-hand-side is bounded (not continuous though). Let $y(0) = 0$. Then if there was a differentiable solution, it would have been decreasing at $t = 0$ and near $t = 0$. Thus $y(t) < 0$ for some $t > 0$. However, for all negative values of y, the solution is increasing (in fact, increasing at a constant rate). This is a contradiction!

So to solve $y' = f(y, t)$, it seems that f better be continuous. Since $y' = \sqrt{y}$ has no uniqueness, perhaps f being differentiable is even better. Going over our Gronwall inequality proof of uniqueness, it seems that the crucial point was $\|Ay\| \leq \|A\|\|y\|$. We can generalise this condition: Let $D \subset \mathbb{R}^n$ be an open connected set. A function $f : D \to \mathbb{R}^m$ is said to be locally Lipschitz if for every $x_0 \in D$, there exists a number M_{x_0} (called a Lipschitz constant) and a neighbourhood N_{x_0} of x_0 such that on N_{x_0}, $\|f(x) - f(y)\| \leq M_{x_0}\|x - y\|$. If the Lipschitz constant is independent of x, it is called Lipschitz.
Indeed, here is a general uniqueness result.

Theorem 1. *There exists at most one differentiable solution $y : [0, h) \to \mathbb{R}^n$ (where h is an extended real) to $y' = f(y, t)$ with $y(0) = y_0$ where $f : D \to \mathbb{R}^n$ is uniformly locally Lipschitz in y (that is, the Lipschitz constant for y is independent of t as well for a neighbourhood of the point) and D is a domain that contains $(y_0, 0)$.*

Proof. Suppose there exist two such solutions y_1, y_2. The set S of all $t \in [0, h)$ on which they coincide is non-empty ($0 \in S$) and closed (why?). If we just prove that it is open, we will be done (why?) So suppose $t_0 \in S$. We shall prove that an interval around t_0 is also contained in S. Indeed, $(y_1 - y_2)' = f(y_1, t) - f(y_2, t)$ and hence $y_1 - y_2 = \int_{t_0}^{t} (f(y_1, s) - f(y_2, s))ds$. Thus $\|y_1 - y_2\| \leq |\int_{t_0}^{t} \|f(y_1, s) - f(y_2, s)\|ds|$. By the uniform local Lipschitzness of f, there exists an ϵ such that on $B_\epsilon(y(t_0)) \times (t_0 - \epsilon, t_0 + \epsilon)$, f is Lipschitz with constant M. Now since y_1, y_2 are continuous, on $(t_0 - \delta, t_0 + \delta)$ (where $\delta < \epsilon$) we have $y_1, y_2 \in B_\epsilon(y(t_0))$. Thus if $t \in (t_0 - \delta, t_0 + \delta)$, $\|y_1 - y_2\| \leq M|\int_{t_0}^{t} \|y_1 - y_2\|ds|$. Let $u = \|y_1 - y_2\|$ and assume that $t > t_0$ WLOg. Then $u \leq M\int_{t_0}^{t} u(s)ds$. By Gronwall's inequality, $u(t) = 0$ on $(t_0 - \delta, t_0 + \delta)$. □

Now what are examples of functions that are uniformly locally Lipschitz in y?

1. $f(y,t) = e^y h(t)$ where h is continuous: Indeed, $|f(y_2,t) - f(y_1,t)| = e^{y_1} h(t) |e^{y_2 - y_1} - 1| \leq C|e^{y_2-y_1} - 1|$ on a neighbourhood of y_1, t. Now $|e^h - 1| \leq C|h|$ for $|h| \leq 1$ for instance.

2. $f(y,t) = \sqrt{y}$ is NOT near $y = 0$: $\frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} \to \infty$ as $y \to 0$.

3. $f(y,t) = |y|h(t)$ where h is continuous.

4. Here is a general example: Suppose $\frac{\partial f}{\partial y}$ exists on the domain and is continuous (jointly in t, y) then indeed, $|f(y_2,t) - f(y_1,t)| \leq \int_{y_1}^{y_2} |\frac{\partial f}{\partial y}| dy \leq C|y_2 - y_1|$ in a neighbourhood of (y_1, t).

5. So if f is compactly supported and smooth, it is Lipschitz (not just locally so).

Now we want to prove an existence result. How does one come up with something from nothing? Well, firstly, if were to solve this problem numerically, we would have tried Euler's method: $\Delta y \approx \Delta t f(y_0, t_0)$, if $y_1 = y_0 + \Delta y$, then $y_2 - y_1 \approx \Delta t f(y_1, t_1)$ and so on. Taking cue from this idea, let us try to solve $y(t) - y_0 = \int_{t_0}^{t} f(y(s), s) ds$ by an iterative method: That is, $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s) ds$ with $y_0(t) = y_0$.